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No. 21

# THE DYNAMICS OF THE AIRPLANE

BY

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NEW YORK

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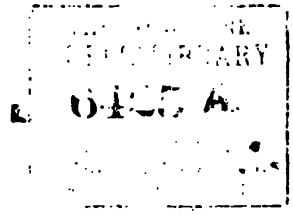

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# PREFACE

IT was the good fortune of the author to attend the University of Paris during the spring semester of 1919. One of the special courses which the French authorities, with their characteristic hospitality, arranged for the large number of students from the American army, was a course in aërodynamics, given by Professor Marchis. The comprehensive knowledge that Professor Marchis possessed of all branches of the new science of aëronautics, the inestimable value of his advice to the French Republic during the war, the interest he took in his rather unusual class, could not fail to be an inspiration.

This book is an outgrowth of those parts of Professor Marchis' lectures that were of particular interest to the author. It is in no sense a complete treatise on aviation. Questions of design and construction are passed over with bare mention. The book is intended for students of mathematics and physics who are attracted by the dynamical aspect of aviation. The problems presented by the motion of an airplane are novel and fascinating. They vary from the most pleasing simplicity to the most stimulating difficulty. The question of stability, particularly, exhibits at the same time the elegance and the power of analysis, and shows the adaptability of some of the general developments in dynamics. The field is assuredly a fruitful one of study, and increasing demands will be put upon the mathematician as the science of aviation continues its rapid development. The mathematician can well own a sense of pride that he had already at hand, in the developments inaugurated by Euler and Routh, a means of dealing accurately with the question of stability, that plays so fundamental a rôle in the science of flying.

The treatment in the text is for the most part elementary. The last chapter alone demands of the student familiarity with more advanced dynamical methods. In the treatment of descent a slight digression is made to consider in part the nature of the solution of a system of two differential equations. This was done in order not to completely evade what seems a problem of considerable difficulty. It might seem that a treatment of the propeller should not find a place in a book with the purpose of this one. No student of mathematics, however, could fail to own a curiosity as to a propeller's action, and it is hoped the discussion, while not complete, will at least serve as a sufficient introduction.

The various curves in the text were plotted by Mr. R. W. Smith, a former student in this university. The author is further indebted to the Smithsonian Institution for permission to use Figs. 12 and 49.

In addition to the various books that are referred to in the text the author has made use of his notes of the lectures of Professor Marchis, translated into English by Madame Ciolkowska, who rendered most valuable aid as an interpreter for those who understood and spoke the language of Professor Marchis only with difficulty.

K. P. WILLIAMS.

Indiana University,
July, 1920.

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# THE DYNAMICS OF THE AIRPLANE

## CHAPTER I

### THE PLANE AND CAMBERED SURFACE

**1.** THE possibility of aërial navigation depends upon the solution of two problems, the problem of sustentation and the problem of propulsion. At the very outset two distinct courses are therefore open. We can look upon the problems as entirely separated from each other, or we can regard them as essentially connected.

In the first case we look for separate solutions, solving first the problem of sustentation, and then, with this successfully disposed of, search for a means of propulsion. This course is historically the older and it is the simpler, for it meets the difficulties one at a time.* A balloon filled with a light

*Early literature abounds with mythical accounts of the flights of legendary heroes equipped birdlike with wings. Among those who seriously studied the question of flight, and actually designed machines with wings to be attached to a person and driven by his own muscular power, Leonardo da Vinci, the renowned artist, occupies the first place. The first instance of people actually ascending from the earth took place November 21, 1783, at Paris. The apparatus was a balloon constructed by Stephen Montgolfier, and the flight covered five miles. The construction of crude dirigibles followed within a few months. The first instance of partial sustentation without the use of gas occurred at this same period. On December 26, 1784, Sebastian Lenormand descended from the tower of the Montpelier Observatory by means of two small parachutes, the idea of the parachute being due to da Vinci. Successful efforts with gliders were made in the last years of the 19th century. Modern aviation dates from 1903, when the Wright brothers first constructed a machine, equipped with engines, which could actually rise from a level field, without the assistance of air currents, and make flights controlled by the pilot. For the history of aëronautics see Albert F. Zahm, "Aërial Navigation," D. Appleton, & Company, New York and London, 1911.

gas, such as hydrogen or helium, affords a means of sustentation. But aërial navigation means far more than the ability to stay aloft, and a craft which can travel only as the wind blows it, can serve few purposes other than that of furnishing amusement. The question of equipping a balloon with engines and a means of propulsion, of traveling in a desired direction with a velocity within our control, of maintaining a desired altitude, of rising and landing, must be answered before we can say the balloon has furnished a means of aërial navigation. It is only since the development of the gasoline motor that this has been possible on any extensive scale.

The second method of attacking the problem seeks to solve simultaneously the problems of sustentation and propulsion. The possibility of propulsion must now come first, for sustentation will be obtained from the motion. It is then evident that this method, even more than that of the balloon, had to await the perfection of a source of energy such as the gasoline engine. It is only with aircraft of this second sort that we are concerned. Such a machine is called an airplane, or aëroplane. We shall adopt the first term. Both names are suggested by the fundamental rôle played by surfaces approximately plane with which the machine is provided. The air reaction on these surfaces, produced by the motion of the machine, furnishes the sustentation. The complete machine will consist of other members, and we can divide it into five distinct parts: the sustaining surfaces, the stabilizing and controlling surfaces, the motor-propeller group, the body, or fuselage, with its place for pilot, passengers and freight, and the landing gear. To these main parts must, of course, be added the various elements of construction by which the different parts are united and the requisite strength given to the complete machine.

We shall not give a discussion of the complete construction of an airplane, but limit ourselves to those features which are necessary for a comprehension of the dynamical problems which we shall study.

**2.** The principles that govern the construction of an airplane, the phenomena that operate during its flight and deter-

mine its behavior, are derived from an understanding of the laws concerning the effect of the wind upon flat and curved surfaces. We can try to determine these laws in two ways, mathematically or experimentally.

In the mathematical method we begin with the principles and equations of hydromechanics. We then see if we can calculate the pressure that a current of air, moving with a certain velocity, will exert upon, for instance, a rectangular plane surface. We must be able to do this for different inclinations of the plane to the air stream. The problem is one of great complexity. In order to construct differential equations that will exhibit the phenomena, and in order to integrate these, we must make assumptions that lead our results to differ from carefully measured observations. For instance, we may assume that the air is a perfect fluid, that is, that it is neither viscous nor compressible. The last assumption seems to be justified for the range of velocities occurring in aëronautical work, but the assumption as to the viscosity vitiates our results when we apply them to actual problems. Even with the assumption of an ideal fluid it is difficult to handle the equations involved. While we can in certain instances obtain information as to how the air streams around obstructing objects, and how it behaves in their vicinity, the results are not such as to make this a simple or satisfactory method of attacking the problem.*

The laws concerning air reaction are determined experimentally in a wind tunnel.† A current of air of several feet thickness is obtained by means of a large fan. Various and accurately known velocities can be given to the air stream. The surfaces upon which we wish to study the pressure are, of course, of limited dimensions, but are similar to those which

* For an elementary mathematical treatment see Cowley and Evans, "Aeronautics in Theory and Experiment," Longmans, 1918, Chapter III.

† Among the various aërodynamical laboratories, can be mentioned those of M. Eiffel at Paris, the National Physical Laboratory of England, the Massachusetts Institute of Technology, and Leland Stanford University. A description of the equipment of such a laboratory can be found in Smithsonian Miscellaneous Collection, Vol. 62.

are to be employed in practice. They are held in the current of air by arms that are connected with balances constructed so as to allow a determination of the magnitude and direction of the air reaction. By changing the shape of the surfaces, the velocity of the air stream, the orientation of the body, etc., the laws which furnish the basis for the design of an airplane, as well as the knowledge of its behavior, are determined. Questions relating to the stability of a machine, to the properties and efficiency of propellers, are also investigated in the same way.

We pass to the consideration of some of the basic aërodynamic laws, as determined empirically.

**3. Pressure on a Plane.**—When a portion of a plane surface is introduced normal to an air current, the pressure produced upon it is a force that tends to displace it, and we find by experiment that the force may be written

$$F = KAV^2,$$

where $F$ is the force, $A$ the area, $V$ the velocity of the air stream, and $K$ a constant for surfaces geometrically similar.* Strictly speaking, $K$ depends upon the size as well as shape, but changes slowly and seems to approach a limiting value as the plate becomes larger. For instance, for a square plate or circle we find, provided $A$ is expressed in square feet, $V$ in miles per hour, $F$ in pounds:

| Side of Square or Diameter of Circle in Feet. | $K$ |
|---|---|
| 0.5 | .00269 |
| 1.0 | .00286 |
| 2.0 | .00314 |
| 3.0 | .00322 |
| 5.0 | .00327 |
| 10.0 | .00327 |

*This law can also be obtained from theoretical considerations. Consider the air as composed of separate particles, moving parallel and striking the plate, normal to their direction of motion. Assume that all particles are brought to rest

We note the slight change produced in $K$ when the side of the square is changed from 3 to 10, although the area is increased over ten times.

The value of $K$ will depend upon the units in which we are expressing $F$, $A$, and $V$. It is necessary to be able to change from one system to another. This question is discussed in the appendix. We shall assume, unless the contrary is stated, that forces are measured in pounds, areas in square feet, and velocities in miles per hour.

The value of $K$ also depends upon the density of the air. The values given above are for a temperature of 0° C., and a pressure of 760 mm. of mercury.

**4. The Inclined Plane.**—Let the plane be inclined at angle $\theta$ to the direction of air flow. This angle is called the **angle of attack.** Neglecting what is called skin friction, it follows that the resultant force is normal to the plane. Its magnitude and point of application vary with $\theta$.* The nature of the air flow about the plane is quite complicated, but can be investigated by photography. By such means information is obtained as to how the air divides in front of the plane, flows over the upper and lower edges, and unites again behind the plane.

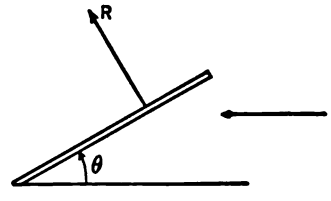

FIG. 1.

by the impact. The pressure on the plate will equal the momentum lost by the air. The quantity of air that comes into contact with the plate in a unit of time is $\rho A V$, where $\rho$ is the density of the air. As all particles lose their velocity $V$, the momentum lost will be $\rho A V^2$. The errors in the hypothesis are apparent, but experiment, while giving a value of the constant different from that deduced by the reasoning above, confirms the qualitative nature of the law.

* If we denote by $F_\theta$ the value of the force for the angle $\theta$ so that $F_{90}$ represents the force for the normal plane, various formulæ exist for obtaining $F_\theta$. Newton gave from theoretical reasons $F_\theta = F_{90} \sin^2 \theta$, which is totally discordant with experiments. The formula of Colonel Duchemin is much more accurate. He gave

$$F_\theta = F_{90} \frac{2 \sin \theta}{1 + \sin^2 \theta}.$$

We are especially interested in the vertical and horizontal components of the total force $F$. We call these components the **Lift, L,** and **Drag, D.** It is proved by experiments that we may write

$$L = K_y A V^2, \qquad D = K_x A V^2,$$

where $K_y$ and $K_x$ are constant for a given angle of attack.

For a square plane we can take the following values:

| Angle. | $K_y$ | $K_x$ |
|---|---|---|
| 5° | .00045 | .00007 |
| 10° | .00097 | .00019 |
| 20° | .00208 | .00074 |
| 30° | .00291 | .00173 |

**5. The Center of Pressure.**—It is important to know the manner in which the point of application of the resultant pressure varies as the angle of attack changes. The general result can be stated as follows:

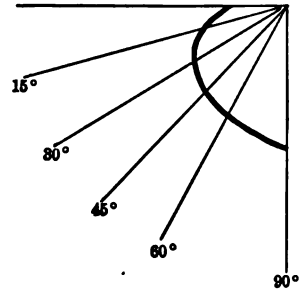

FIG. 2.

*As the angle of attack diminishes the center of pressure approaches the leading edge.*

This behavior of the center of pressure is shown in Fig. 3.

A knowledge of the movement of the center of pressure is of importance in the subject of stability and in finding the forces on the control surfaces.

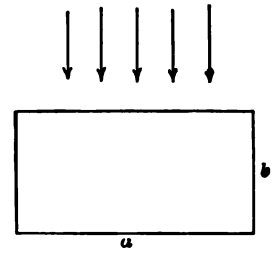

FIG. 3.

**6. Aspect Ratio.**—It was stated above that the quantity $K$ in the fundamental equation for the pressure was constant for planes geometrically similar. In case we have a rectangle of sides $a$ and $b$, situated, as shown, in an air current, we call the fraction $a/b$ the **aspect ratio.** This quantity is of importance. We find that the coefficients $K$, $K_y$, $K_x$, which have been used

above, vary when the aspect ratio is changed. Those values which have been given for a square may, however, be used with fairly accurate results for planes with aspect ratio close to unity. The dependence of the quantities $K$, $K_y$, $K_z$ on the aspect ratio comes from the important effect of the boundary of the surface upon the resistance. If we have a long narrow rectangle, it is evident that the escape of the air around the boundary will greatly alter the pressure from what it would be for the equivalent square plane.

**7. The Cambered Wing.**—If we examine a bird wing, we find that it is not flat, but is curved. This suggests that there may be some aërodynamic advantage in such a surface. Experiment amply confirms this, and the sustaining surfaces of all airplanes are curved, or **cambered**. Evidently also the surface must have thickness for constructional reasons.

The word **aërofoil** is used to designate a sustaining surface or wing. In Fig. 4 there is illustrated the general shape of the section of an ordinary aërofoil. We call $AB$ the **chord**, $DC$ the **camber** of the upper surface, $EC$ the **camber** of the lower surface, $C$ the position of maximum camber. The quantities $BC$, $CD$, $CE$ are generally expressed in terms of the chord $AB$.

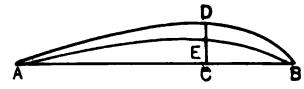

FIG. 4.

Let the wing be placed with reference to the air flow as shown in Fig. 5. Then by the **angle of attack** is meant the angle $\theta$ between the chord and the direction of the relative wind. Let $N$ represent the direction of the normal and $R$ the resultant pressure. It is found that for small angles $R$ is in advance of $N$. This increases the lift $L$, and diminishes the drag $D$. These are desirable effects, and it is partly on account of this property that the cambered wing is more efficient for sustaining purposes than the plane wing.

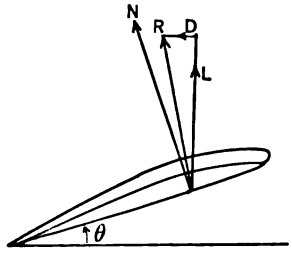

FIG. 5.

It is evident what we mean by the terms **leading edge, trailing edge,** and **nose.**

**8. Characteristic Curves for a Given Wing.**—By means of experiments in a wind tunnel we investigate $R$, $L$, $D$ as functions of $V$ and $\theta$. We find that we can write, as for a flat plane,

$$R=KAV^2, \qquad L=K_yAV^2, \qquad D=K_xAV^2,$$

where

$$K^2=K_y^2+K_x^2.$$

The quantities $K$, $K_y$, and $K_x$ are again constants for a given angle and geometrically similar aërofoils. The values of the coefficients $K_y$ and $K_x$, and the ratio $L/D=K_y/K_x$ for a certain aërofoil * are given in the following table. The table also gives the position of the center of pressure, which is considered in § 12.

| Angle of Atack. | $K_y$ | $K_x$ | $L/D$ | Distance of C. P. from Leading Edge |
|---|---|---|---|---|
| $-4°$ | $-.000399$ | .0001515 | 2.64 | |
| $-2$ | $+.000156$ | .0000905 | 1.72 | |
| $-1$ | .000432 | .0000700 | 6.15 | .620 |
| 0 | .000721 | .0000653 | 11.00 | .530 |
| 1 | .000936 | .0000670 | 14.00 | .463 |
| 2 | .001146 | .0000688 | 16.60 | .415 |
| 4 | .001510 | .0000860 | 17.50 | .340 |
| 6 | .001878 | .0001158 | 16.20 | .316 |
| 8 | .002230 | .0001558 | 14.30 | .303 |
| 10 | .002580 | .0002055 | 12.60 | .290 |
| 12 | .002910 | .0002595 | 11.20 | .283 |
| 14 | .003165 | .0003040 | 10.40 | .274 |
| 16 | .003165 | .0003710 | 8.50 | .276 |
| 18 | .003080 | .0005520 | 5.60 | .310 |
| 20 | .002882 | .0008500 | 3.40 | .360 |

In order to use these values to determine the lift and drag, the velocity $V$ must be given in miles per hour, the area $A$ of the wing in square feet. The values of $L$ and $D$ that are then given by the formula will be in pounds.

* This wing is U. S. A. No. 1 in the Third Annual Report of the (American) Advisory Committee on Aëronautics. The shape of the wing is considered in § 10.

The values of $K_y$, $K_x$ and the ratio $L/D$ are also plotted in the following curves. It is to be noted that the vertical scale is not the same for the three different quantities.

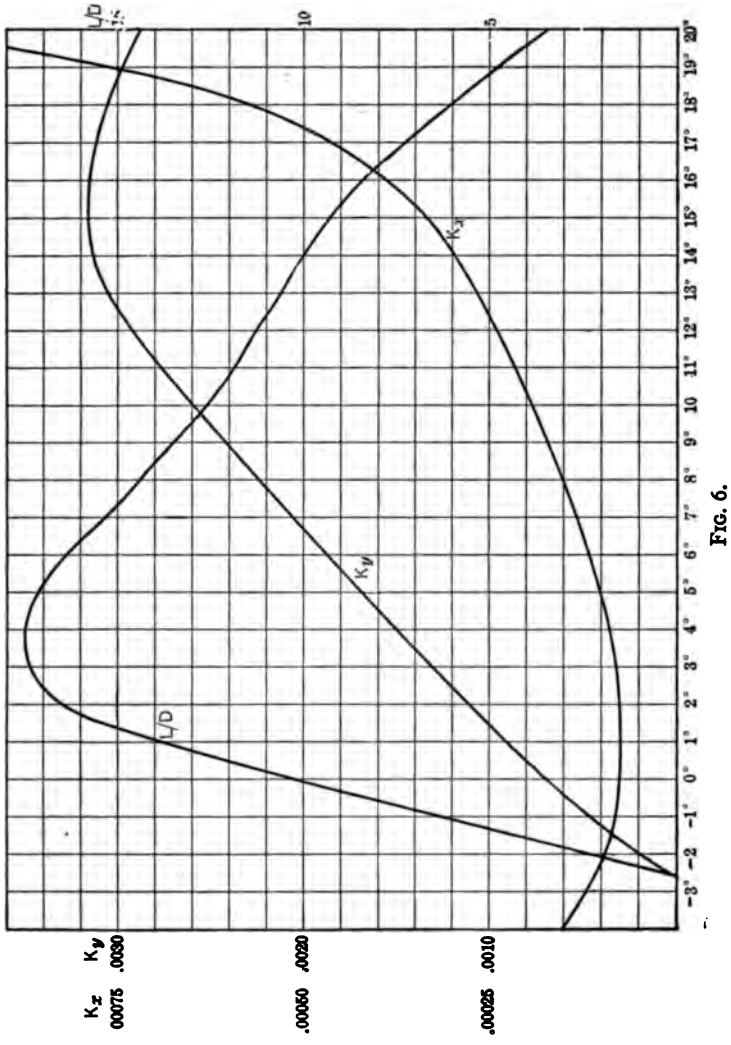


FIG. 6.

We note the following facts:

1. *There is lift at an angle $-2°$, i.e., sustentation exists for a negative angle of attack.*

*2. The lift increases almost as a linear function of the angle and attains a maximum at about 15°, then decreases rapidly.*

*3. The drag remains sensibly the same over small angles and increases very abruptly in the vicinity of 15°.*

*4. The ratio L/D increases practically as a linear function for small angles, and attains a maximum in the vicinity of 15°.*

We shall merely note here the importance of the ratio $L/D$. For horizontal flight the lift must equal the weight of the machine. Consequently the greater the quantity $L/D$ the

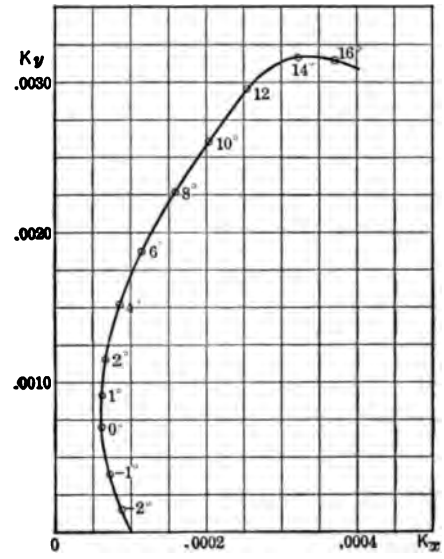


FIG. 7.

less resistance there is to be overcome, and consequently the less power is necessary for a given speed of flight.

It is from a study of the characteristic curves for a given aërofoil that one decides upon its efficiency or suitability for a given type of machine. It is not our purpose to go into this question, and we shall merely remark that the type of wing to be selected depends upon whether the machine is designed for great speed, for rapid climbing, or for carrying heavy loads.

**9. Polar Diagram.**—Instead of plotting the lift and drag coefficients with the angle of attack as argument, we can plot

the lift against the drag. It gives, however, better results if if we take different scales for $K_y$ and $K_x$. We obtain in this way the polar diagram, which has been extensively used by M. Eiffel. In what follows we shall see its suitability for many purposes.

**10.** The section of any wing depends upon the values of $BC$, $DC$, $EC$, measured in terms of the chord $AB$, upon the shape of the nose, and the general shape of the trailing edge. Questions of strength and facility of construction are intimately connected with those of thickness. By varying the different elements one at a time, we are able to arrive at conclusions as to the best value of any element, and determine the best section for a given purpose. As an average we can state that $CB$ equal 3/8, and $CD$ lies between 0.05 and 0.08. The effect of the under camber is not so well known.

In Fig. 8 there is given the shape of the wing for which the curves are given in § 8.

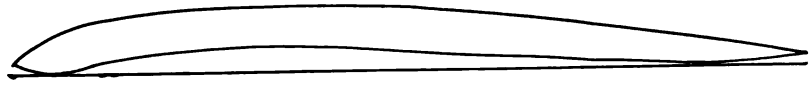

FIG. 8.

It is also necessary to study the effect of the shape of the ends of the wing. At the ends leakage occurs, and the air flow is accordingly greatly modified. A wing with trailing edge slightly longer than leading edge is found to be most efficient.

For structural reasons the aspect ratio does not usually exceed 8, as the advantage secured from increased lift is then overbalanced by the increased weight of construction.

**11. Pressure over the Wing.**—It is not only possible to study the total resultant pressure on a wing, but also by introducing tubes through small holes in the surface, and connecting them to a manometer, it is possible to determine the pressure at different points. The results for a central section are shown in Fig. 9. It is found that the pressures on top of the wing are below atmospheric. Consequently there is

suction on top of the wing. The pressures underneath the wing are greater than atmospheric, so there is an active upward pressure. The two combine to make the total upward pressure. In the figure, the suction on the top surface is represented by lines drawn outward from it, and the pressure on the lower surface, similarly by lines drawn outward. It is seen that the suction is the greater of the two forces. In fact, it contributes about three-fourths of the total sustaining force. This also shows why there can remain sustentation for negative incidence.

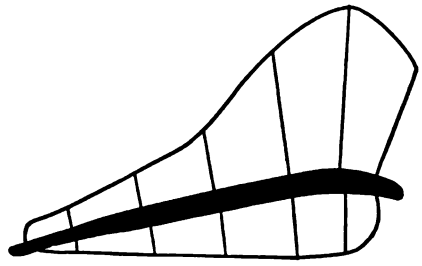

FIG. 9.

We find also in this way that we must not make the chord too great, or towards the trailing edge we have pressures above, and suction below, which would lessen the sustentation.

In the same way the pressures along any section can be studied, so that the pressures all over the surface can be mapped, and we obtain a very clear visualization of the air reaction.

It should be noted here that the difference of pressures from atmospheric pressure are very slight. But by having large wing surfaces, and obtaining sufficient velocity we can make the total lift equal to or greater than the weight of the machine.

**12. Center of Pressure.**—The resultant pressure is a vector, and so is completely fixed in magnitude and direction. We know all there is to know about it if we know its magnitude and its moment about some point, for instance the leading edge. It is, however, customary to speak of the **center of**

**pressure,** which we define as the point where the vector representing the pressure intersects the chord.

It is important for us to know how the center of pressure behaves as the angle of attack changes. In general, it is found for cambered surfaces and the angles employed in aviation that the following is true:

*The center of pressure recedes from the edge of attack as the angle of attack diminishes.* For larger and increasing angles it recedes. This property should be contrasted with that given above for plane surfaces.

Fig. 10 shows the motion of the center of pressure for the wing considered in § 8.

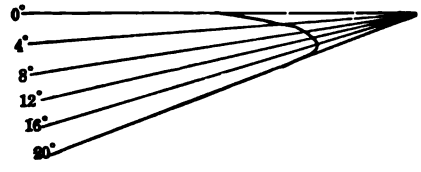


FIG. 10.

**13. Relation between $K_y$ and $K_x$.**—It is impossible to obtain a simple relation between the coefficients $K_y$ and $K_x$, but an approximate one will be obtained, and use will be made of it later.

In the note to § 4 there is given the formula of Colonel Duchemin for the pressure on a flat plane as a function of the angle of attack. It is obvious that for small angles it varies approximately as the sine of the angle. Let us assume such a relation for a cambered wing. As there is still lift for a negative incidence we would expect to measure angles from the position of the wing that gives no lift. We shall not do this, however, and the results obtained will not be valid for the small angles of attack.

We assume a relation

$$K = K' \sin \theta,$$

where $K$ is the coefficient of pressure, and $K'$ a constant.

Assuming that the pressure is normal to the chord we would then have

$$K_y = K' \sin\theta \cos\theta, \qquad K_x = K' \sin^2\theta.$$

Hence

$$\frac{K_x}{K_y^2} = \frac{1}{K' \cos^2\theta}.$$

If $\theta$ is small this is approximately constant.

In order to see something about the accuracy of our result we shall actually calculate the value of $K_x/K_y^2$ for the values given in § 8. The results are as follows:

| Angle | 1° | 2° | 4° | 6° | 8° | 10° | 12° | 14° |
|---|---|---|---|---|---|---|---|---|
| $K_x/K_y^2$ | 76.40 | 52.3 | 47.4 | 33.6 | 31.3 | 30.8 | 30.6 | 30.3 |

While the value is not constant, it is seen that between 6° and 14° it is practically so, and approximate results can be obtained by assuming that it is constant.

**14. The Biplane.**—In order to secure greater lifting forces, and at the same time deal easily with the problem of construction, it is customary to use two or more surfaces, one above another. We shall limit ourselves to the biplane.

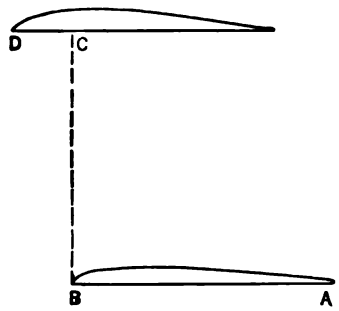

FIG. 11.

By the **gap** is meant the distance between the two planes, measured in terms of the chord, i.e., the ratio $BC/AB$. By the **stagger** is meant the distance $CD$, also measured in terms of $AB$. The stagger is positive if the upper plane is in advance of the lower, negative if in the rear.

The question of the relative efficiency of the upper and lower wings enters at once. The gap is limited by constructional reasons, and usually varies between 1 and 1.1. It is found then that the efficiency of the lower wing is considerably lessened by the presence of the upper wing. The reason becomes apparent when we recall what was said in § 11. The region of

depression above the lower wing is considerably modified by the upper aërofoil. And as this depression is what gives the preponderant part of the lift we would expect the lift to be lessened. On the other hand the suction above the upper wing is unimpaired, while the less important pressure below is somewhat modified. The lower wing then has less efficiency than the upper one, and on this account its aspect ratio is sometimes made smaller than that of the upper wing.

The center of pressure for a biplane is purely a matter of definition. We shall understand it to mean the point of intersection of the vector representing the pressure with a line parallel to the chords and midway between them.

**15. Body Resistance.**—In the forward movement of an airplane, all parts of the machine contribute to the resistance that must be overcome, while the aërofoils alone contribute to the sustentation.* It is necessary to make the resistance of all parts of the machine as small as possible. This is done by giving proper shapes to fuselage, struts, wires, landing gear, etc. The resistance that arises from the parts other than the wings can be conceived of as arising from the motion of a square plane normal to the direction of motion. We call this the equivalent **detrimental surface.** Let its area be $s$; the resistance due to the body can then be written

$$f = ksV^2.$$

For the complete machine we therefore have

$$L = K_y A V^2, \qquad D = K_x A V^2 + ksV^2.$$

**16. Experiments on Complete Model.**—We have thus far discussed experiments made on the models of the wings. Models of complete machines, with the exception of propeller, wires, etc., whose contribution to the complete resistance is small, are also subjected to exhaustive study in the wind tunnel. Data necessary for the discussion of stability are obtained in

* The body doubtless adds something to the sustentation, but it is not an amount of which we can take account, and is negligible when compared with that furnished by the wings.

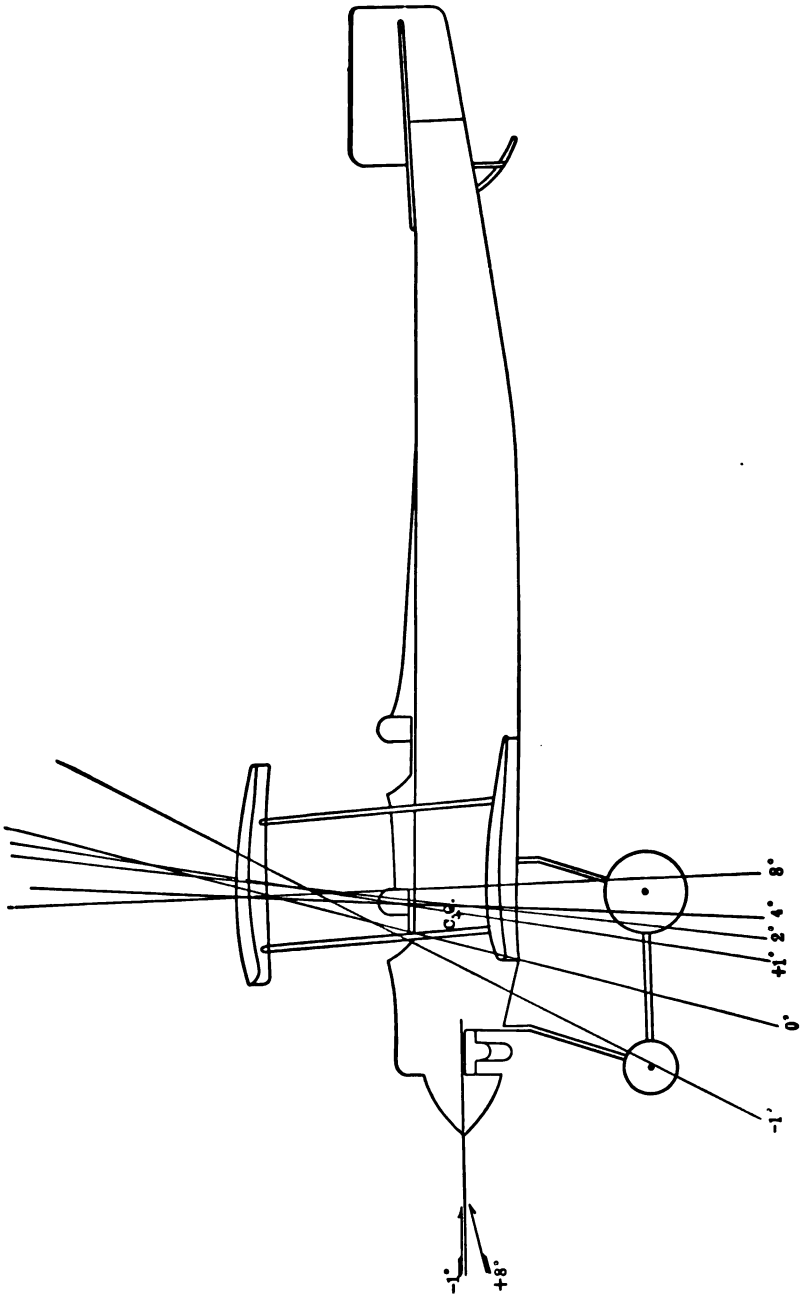


FIG. 12.

this way. A diagram of a model of this sort is shown in Fig. 12.* Lines showing the direction of the air reaction for angles of attack varying from $-1°$ to $8°$ are shown.

A question of primary importance at once arises. To what extent, and under what conditions are the results obtained from experiments on models applicable to full-scale machines? The study of this question depends upon what is called dynamical similarity. We shall not touch upon it here. In the appendix reference will be given to treatments of this important question.

* This is taken from Hunsaker's paper, "Dynamical Stability of Aëroplanes," Smithsonian Miscellaneous Collections, Vol. 62, No. 5, 1916. It is a model of a biplane tractor designed by Captain V. E. Clark, U. S. A., and is representative of modern design.

## CHAPTER II

### STRAIGHT HORIZONTAL FLIGHT

**17.** Let us consider the motion of an airplane. Suppose the motor is running. The machine is at any instant acted upon by three forces, its weight, acting downwards through the center of gravity, the traction of the propeller, and the resistance of the air. As the machine is a rigid body we can in general combine these three forces into a single force and a couple. The force determines the instantaneous motion of the center of gravity, and the couple determines the rotation which the machine is momentarily undergoing. When regarded in this general way the problem is seen to be very complex, on account of the nature of the resistance of the air, for this resistance depends upon the angle at which the wings are any instant attacking the air, and on the velocity. Thus both the motion of the center of gravity and the rotational motion affect it.

In order to fix the ideas, suppose the airplane moves constantly in a vertical plane, coincident with the plane of symmetry of the machine; then the center of gravity traces out a certain plane trajectory. Suppose further that the angle of attack remains constant; the angle between the tangent to the trajectory and the chord of the wings is then constant. The relative wind at any instant is in the direction of the tangent. Furthermore, it is evident that what we have called the lift in the preceding chapter will now be in the direction of the normal to the trajectory, and the drag will be in the direction of the tangent. The motion of the machine will then be determined by a force equal to the weight downward through

the center of gravity, the traction of the propeller along the tangent to the path, the lift

$$K_y A V^2$$

perpendicular to the tangent, and the combined drag of the wings and body

$$K_x A V^2 + ksV^2$$

along the tangent to the path.

**18. Horizontal Flight.**—We consider first the simplest possible case. Let the machine be moving horizontally in a straight line, with constant angle of attack. What are the conditions that must be fulfilled to make this possible, and what properties about the motion can be discovered?

The lift $L$ is now a vertical force, and the drag a horizontal one. In the figure, $F$ represents the resultant air pressure, applied at $P$, $W$ represents the weight applied at the center of gravity, and $T$ the traction of the propeller applied at a point $B$, any point in the axis of the propeller.

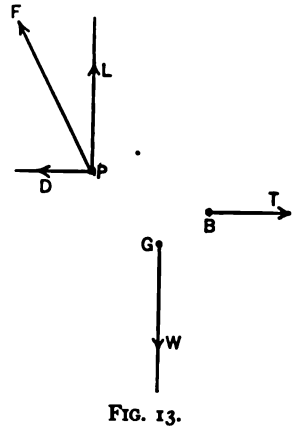

FIG. 13.

We are supposing that the machine is moving horizontally. Hence $W=L$. Furthermore, we must have $T=D$, for if, for instance, $T>D$, the velocity would start to increase. This would increase $L$, and the machine would start to rise. Likewise, if $T<D$, the machine would start downwards. Hence, at a constant angle of attack, uniform motion alone is possible, if the machine is to fly horizontally.

In order that there be no rotation, the couple formed by $W$ and $L$ must be equal to that formed by $D$ and $T$, and have the opposite sense. Consequently the three forces, $F$, $T$, and $W$ must be concurrent.

The equations of motion are therefore:

$$W = K_y A V^2,$$
$$T = K_x A V^2 + ksV^2.$$

**19. The Velocity.**—The first of these equations is called the **equation of sustentation.** From it we find the velocity

$$V = \sqrt{\frac{W}{A} \cdot \frac{1}{K_y}},$$

with which the machine must move to produce sustentation. From the manner in which $K_y$ varies with the angle of attack, we see that the smaller the angle of attack is, the greater must be the velocity.* Furthermore, the velocity increases with the quantity $W/A$. This quantity is the weight per unit area, or the **loading**, and we see the importance that it plays. It is not the weight, but the loading, that determines the speed of the plane for a given angle of attack.

**20. Landing Speed.**—It is apparent that landing must be accomplished by flying horizontally very close to the ground at the smallest velocity possible. This will be with the angle of attack giving the greatest value of $K_y$. If we have chosen the type of wing, and the weight of the machine, we can obtain the area of the wing by fixing the landing speed.

Suppose we wish to construct a monoplane weighing 1200 pounds, the curves for the wing being those in § 8. Let the landing speed be 45 miles per hour. The maximum lift coefficient available we take as 0.00316, corresponding to an angle of attack of about 14°. We have then

$$A = \frac{1200}{.00316 \times 45^2} = 187.5 \text{ square feet.}$$

This gives us a loading of 6.40 pounds per square foot.

The value of the velocity of the machine for different angles of attack is best visualized by a curve. Using the loading

*The means employed to cause the machine to fly at different angles of attack is considered in Chap. VII, where controllability is discussed. See also § 30.

that we have just found and the values of $K_y$ from § 8 we obtain the curve given in Fig. 14.

**21. Effect of Altitude.**—With the same angle of attack a machine will not fly horizontally at different altitudes with the same velocity, on account of the change in the density of the air as we ascend. The coefficients $K_y$, $K_x$ and $k$ all vary directly as the density of the air. Thus, if we let $K_y(z)$, $K_x(z)$, $k(z)$ represent the values at altitude $z$, and $K_y$, $K_x$, $k$ the values

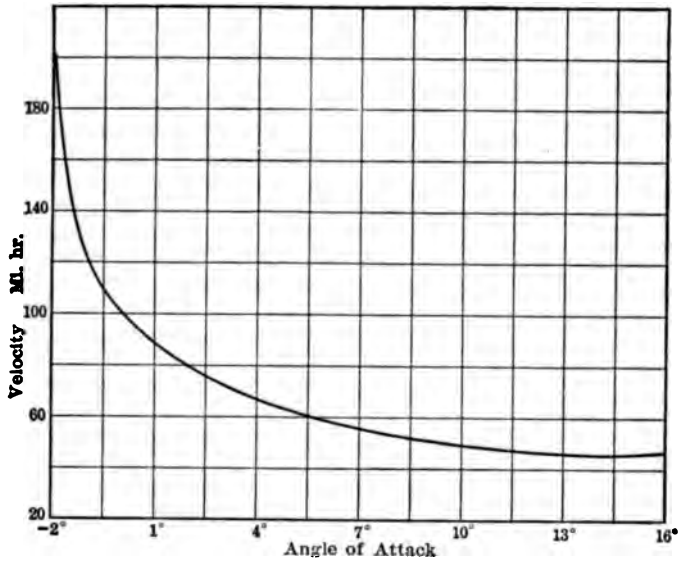


FIG. 14.

at the surface of the earth, which are the values actually given for any wing, we have

$$\frac{K_y(z)}{K_y} = \frac{K_x(z)}{K_x} = \frac{k(z)}{k} = \rho(z),$$

where $\rho(z)$ denotes the ratio of the density of the air at the altitude $z$ to the density at the surface of the earth. The value of the decreasing function $\rho(z)$ is given in a table in the Appendix.

It is evident that for a given angle of attack the velocity of the machine varies inversely as the square root of the density

of the air. For a given angle of attack the machine therefore flies faster as the altitude increases.*

**22. The Effort of Traction.**—The traction that the propeller must furnish is given by the second equation in § 18:

$$T = K_x A V^2 + ksV^2.$$

If we take the quotient of $T$ and $W$ we obtain

$$\frac{T}{W} = \frac{K_x + k \cdot s/A}{K_y}.$$

It is evident that the right-hand side is independent of the altitude, for $K_y$, $K_x$, $k$ are all multiplied by the same quantity as the altitude increases. *The traction, therefore, does not depend upon the altitude, but only on the angle of the attack.*

**23. Optimum Angle.**—Consider the polar curve given in § 9. Suppose for a moment that the unit used along the two axes is the same. Take a point $Q$ at distance $ks/A$ to the left of the origin. Let $P$ be a point on the curve corresponding to an angle of attack $\alpha$. Then it is evident that

$$\frac{T}{W} = \tan \phi,$$

where $\phi$ is the angle that $QP$ makes with the $K_y$-axis. It is seen that $QP$ will, in general, intersect the polar in another point $P'$, corresponding to a different angle of attack, $\alpha'$. But the speeds necessary for sustentation at angles of attack $\alpha$ and $\alpha'$ are different. We can therefore fly at two different speeds, and experience the same resistance, and thus require the same traction from the propeller.

The effort of traction will be a minimum when $\tan \phi$ is a minimum, and this occurs when $QP$ is tangent to the polar. Let $M_1$ be the point of tangency, and $\alpha_1$ be the corresponding angle of attack. The angle $\alpha_1$ is called the **optimum angle.**

* We really should say, the machine must be made to fly faster in order to produce sustentation. What requirement this puts on the power that the motor is furnishing will be considered later.

In case we have used different horizontal and vertical scales on the polar, as is convenient, we can no longer obtain $T/W$ by merely taking tan $\phi$. However, we shall still obtain $\alpha_1$ by taking the point $M_1$ such that $QM_1$ is tangent to the polar. We can also find the second angle of attack for which the traction is the same as that for some given angle, by finding as before, the second point of intersection of $QP$ and the polar. The distance $OQ$ must be plotted in the same unit used for $K_x$.

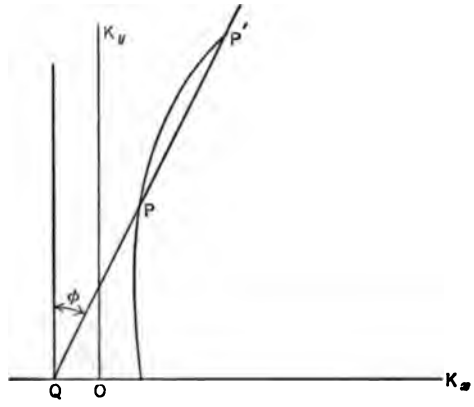


FIG. 15.

**24. Fineness.**—The value of the ratio $T/W$ is called the **fineness** for a given angle of attack. It is easily calculated for a machine when we know $K_y$, $K_x$, $s$, and $A$. The importance that it plays in the performance of the machine will appear as we proceed. In certain respects the fineness completely characterizes the machine. It is desirable that the fineness be as small as possible. For a given value of the fineness the traction necessary for sustentation varies directly as the weight of the machine. The minimum value of $T/W$ is sometimes spoken of as the fineness of the machine.

Let us assume a detrimental surface of 8 square feet, and take $A = 187.5$, as found in § 20. From § 3 we have $k = .00322$. This gives $ks/A = .0001373$. The value of the fineness for the airplane considered is then found to be that given in the following table. We denote it by the letter $B$. The traction $T$ is also given, for the weight $W = 1200$.

| Angle of Attack. | Fineness $B$. | Traction $T$, in lbs. |
|---|---|---|
| $-1°$ | .604 | 729 |
| 0 | .281 | 337 |
| 1 | .218 | 261 |
| 2 | .179 | 216 |
| 4 | .147 | 176 |
| 6 | .135 | 161 |
| 8 | .131 | 157 |
| 10 | .133 | 159 |
| 12 | .136 | 163 |
| 14 | .139 | 167 |
| 16 | .160 | 192 |

From the results given in the table it is possible to plot the traction against the angle of attack. From the curve which is derived and the velocity curve in Fig. 14, it is then possible to plot the traction against the velocity. Such a curve will be given in the same diagram with the power, Fig. 16.

**25.** It is useful at times to consider separately the resistances that arise from the wings and the body of the machine, respectively. It is apparent that curves representing these quantities will be obtained by plotting $K_x A V^2$ and $ksV^2$ against the velocity.

An application of the work of § 13 can be made in connection with the determination of the optimum angle. We desire the minimum of the quantity

$$K_x A V^2 + ksV^2,$$

considered as a function of the angle of attack. The product of the two terms is

$$ksAK_xV^4.$$

Inserting the value of $V$ derived from the equation of sustentation this becomes

$$\frac{ksW^2}{A}\frac{K_x}{K_y^2}.$$

The coefficient of $K_x/K^2_y$ is a constant, and from § 13 it follows that $K_x/K_y^2$ itself is practically constant in the vicinity of the optimum angle. Now from the principles of maxima and minima we know that the sum of two quantities whose product is constant attains its least value when the two quantities are equal. Accordingly, it follows, with a sufficient approximation, that the optimum angle is the angle for which the drag of the wings equals that of the body of the machine. Therefore if we plot the drag of the wings and of the machine separately against the velocity, the optimum angle will be the angle that corresponds to the point where the two curves cross, and the velocity for the optimum angle can be obtained at once.

**26. The Useful Power.**—The useful power developed by the motor is $TV$, which can be written

$$P = K_x A V^3 + ksV^3$$

$$= \frac{W^{3/2}}{A^{1/2}} \cdot \frac{1}{K_y^{1/2}} \cdot \frac{K_x + ks/A}{K_y},$$

if we desire to put the fineness in evidence.

The curve of the useful power necessary is easily obtained from that of the traction by multiplying the ordinate by the abscissa.* The traction and useful power, plotted against velocity, are given in Fig. 16.

**27.** In addition to overcoming the resistance and imparting the velocity necessary for sustentation, the propeller imparts a velocity to the air with which it comes in contact. The force that is expended in this way is equal to the momentum imparted to the air. The resultant energy is lost. The only part of the total work the motor is developing that is useful to us is what is considered above. We can calculate from the characteristic curves of the machine the useful power which

* To obtain the power we may express $V$ in feet per second. The product $TV$ then gives the power in foot-pound-seconds. To change to horsepower divide by 550. If $V$ is in miles per hour and $T$ in pounds we obtain horsepower by dividing the product $TV$ by 375.

we must have for each angle of attack. The total power which the engine must be furnishing is then dependent on the efficiency of the propeller. This question will be considered in its proper place.

**28. The Economical Angle.**—Just as there is an angle of attack for which the effort of traction is a minimum, so there

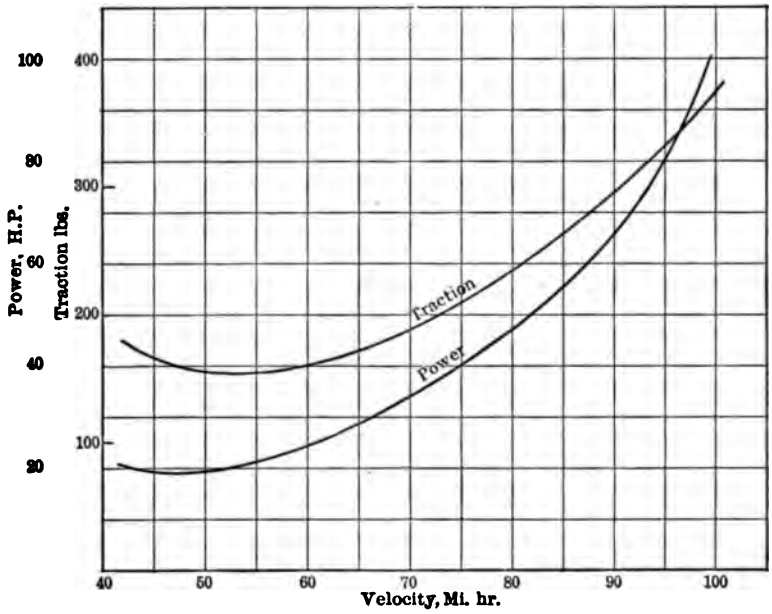


FIG. 16.

is an angle for which the power necessary is the least possible. This angle is not the same as the optimum angle. It is evident at once that it must then be greater. For consider the product $TV$. As the angle of attack decreases from the optimum angle $\alpha_1$ both $T$ and $V$ increase. Hence $P$ increases continuously and could not attain a minimum. As the angle increases from $\alpha_1$ the traction $T$ increases and $V$ decreases, so the product $TV$ may have a minimum. Furthermore, we see that in order that the product $TV$ should have a minimum coincidently with $T$, it is necessary that either $V$ have a minimum at the same time, or that $T$ vanish at its minimum, neither of

which conditions is fulfilled. Hence the power will reach a minimum at an angle of attack greater than the optimum angle.

The angle for minimum power can be found from the true polar, that is, the polar constructed with equal units along the two axes. Take $Q$ as origin and let $x$ and $y$ be the Cartesian coordinates of a point on the polar. Then from § 24 the power can be written

$$P = hx/y^{3/2},$$

where $h$ is constant. To find the minimum of this we equate its derivative to zero, giving

$$\frac{dx}{dy} = \frac{3x}{2y}.$$

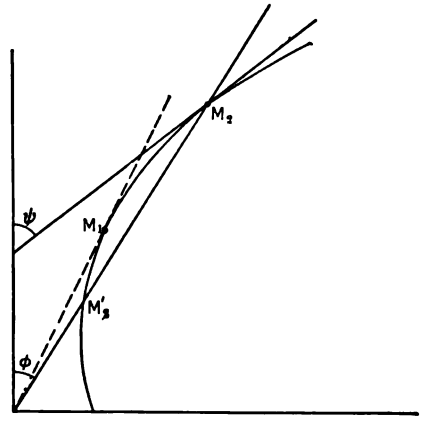


FIG. 17.

Suppose the tangent at any point of the polar curve makes an angle $\psi$ with the $y$ axis. Then the condition just found gives for the point of minimum power

$$\tan \psi = \tfrac{3}{2} \tan \phi,$$

where $\phi$ has the meaning given to it in § 23. At the optimum angle we have $\psi = \phi$ and from the shape of the polar curve we see that the point $M_2$ satisfying the last condition is to

the right of $M_1$, the point corresponding to the optimum angle. The corresponding angle of attack, which we shall denote by $\alpha_2$, is called the **economical angle.** Flight at that angle requires a minimum rate of consumption of fuel, assuming that the efficiency of the propeller is practically constant over the region considered.

It is not difficult to show that the equation last written will also be true at the points $M_2$ and $M_1$ if the polar is constructed with different scales along $K_y$ and $K_x$.

We can summarize the results of this section as follows:

*The* economical angle *is greater than the* optimum angle.

*The machine flies faster at the optimum angle than at the economical angle.*

**29.** Although the consumption of fuel is at a less rate at the economical angle, the machine flies more slowly, so consumption takes place over a longer period, in case a given distance is to be traversed. In fact the work per unit distance traversed is exactly equal to the traction, and is therefore a minimum when the traction is least. Furthermore, high speed is usually a desirable feature, so flight at the optimum angle is preferable to flight at the economical angle.

We can obtain some interesting results by comparing the increased speed with the increased power necessary when flying at an angle other than the economical one.

Let $V_2$, $P_2$ represent, respectively, the velocity and power for the economical angle, point $M_2$ on the polar, Fig. 17, $V$ and $P$ the corresponding quantities for any other angle, represented by $M$ on the polar, $K_{y_1}$ and $K_y$, the corresponding lift coefficients. Then

$$\frac{V}{V_2}=\sqrt{\frac{K_{y_1}}{K_y}}, \qquad \frac{P}{P_2}=\sqrt{\frac{K_{y_1}}{K_y}}\cdot\frac{\tan\phi}{\tan\phi_1}.$$

where $\phi_1$ and $\phi$ are the angles in the polar diagram referring to the conditions of flight. We can then write

$$\frac{V}{V_2}=\frac{P}{P_2}\cdot\left\{\frac{\tan\phi_2}{\tan\phi}\right\}.$$

Let $M_2'$ be the other point where $OM_2$ intersects the polar, and $\alpha_2'$ the corresponding angle of attack. Then if $M$ lies between $M_2$ and $M_2'$ the quantity $\tan \phi_1/\tan \phi$ is greater than unity. Consequently, for angles of attack between $\alpha_2$ and $\alpha_2'$ the speed increases at a greater rate than the power necessary. It is not until the angle of attack falls below $\alpha_2'$ that the increased speed is obtained by a disproportionate increase in necessary power, and therefore fuel.* We see also that the farther $M_2$ is from $M_1$, the point representing the optimum angle, the greater will be the range in values for the angle of attack, for which the increase of speed is not accompanied by a too great increase of power.

**30.** An application of the results that have been obtained can be made to the question of control of the machine. For a given angle of attack the machine can fly horizontally with but one velocity. Alteration in velocity must therefore be secured by changing the angle of attack. This is done by means of the elevator, whose effect, as explained in § 94, is to rotate the machine as a whole about an axis perpendicular to the plane of symmetry. It is apparent that there is a single angle of attack for which the axis of the propeller is horizontal. But the vertical component of the propeller thrust is in all instances small, and it has been neglected. As the power necessary varies with the speed (or angle of attack), a manipulation of the elevator for the purpose of altering the speed must be accompanied by a change in the admission of gasoline to the motor. Otherwise the machine would start to ascend or descend.

We have considered only the useful power necessary to maintain flight at different velocities. The total power the motor is developing is determined by the number of revolutions it is making per minute. The quotient of the useful power and the total power being developed by the motor is the efficiency of the propeller. This is not constant, but depends upon both forward velocity and number of revolutions per minute. The question is discussed more at length in Chapter V. It is

* The statement about the increase of fuel is only approximately correct, as the efficiency of the propeller must be considered.

apparent, however, from what has been said, that as the pilot simultaneously changes the elevator and the admission of gasoline in order to secure a different horizontal velocity, the efficiency of the propeller may be either increasing or decreasing. We shall, however, assume that an increase of useful power will call for an increase of total power, and therefore require an increase in the admission of gasoline. Conversely, we shall assume that a decrease of useful power will be secured through a decrease in the amount of gasoline admitted to the motor.

Suppose the machine is flying at an angle of attack less than the economical angle. To fly faster the pilot must, as he properly manipulates the elevator, admit more gasoline to the motor. Conversely, if he desires to fly slower, he must cut down on the admission of gasoline. If, however, the machine is flying at an angle of attack greater than the economical angle, there is a reversal in the use of the controls. Greater speed will be secured by lessening the admission of gasoline, as the elevator is turned so as to decrease the angle of attack, while more gasoline must be admitted if it is desired to fly slower. If the machine is flying at the economical angle, it will require more gasoline to fly either faster or slower.

# CHAPTER III

## DESCENT AND ASCENT

### 1. Descent

**31.** In the last chapter we have discussed the conditions of rectilinear horizontal flight, considering such questions as velocity, traction, and power. The results form the basis for many problems connected with design and performance of a machine. Other conditions of flight must, however, be considered, and we shall first take up that of descent. Descent, of course, can be accomplished in numerous ways by the proper manipulation of the controls. There are, however, certain conditions of descent of a more definite character, which give rise to some interesting questions.

**32. Equations of Motion.**—Consider a part of the trajectory, starting at a point $P$. Let $G$ be the position of the center of gravity at a certain instant, and $\theta$ the angle that the tangent makes with the horizontal. Since the lift and drag are normal to and along the tangent, respectively, we have for the total forces in these two directions,

$$W \cos \theta - K_y(z) A V^2,$$

$$T + W \sin \theta - (K_x(z) A + k(z) s) V^2$$

where $T$ is the traction, and $z$ is the altitude of the machine.

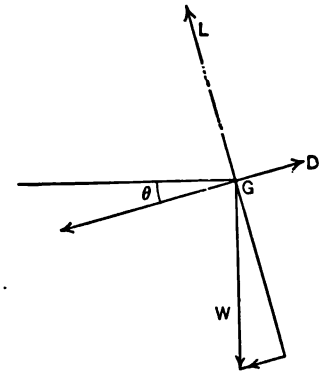


FIG. 18.

The dependence of the various coefficients on the air density has been put in evidence because we shall no longer be con-

sidering the machine at a constant altitude. We are making the assumption that the axis of the machine will maintain itself along the tangent, so that the angle of attack remains constant. If the trajectory is curved this will necessitate a rotation of the machine about an axis perpendicular to the plane of symmetry. The stabilizing surfaces of the machine provide for such a behavior.

From analytical mechanics we know that the normal acceleration of a particle is given by $V^2/r$, where $r$ is the radius of curvature of the path, while the tangential acceleration is given by $\frac{d^2s}{dt^2}$, where $s$ is the length of arc measured from a convenient point. But we can write

$$\frac{V^2}{r}=V^2\frac{d\theta}{ds},\quad \frac{d^2s}{dt^2}=\frac{dV}{dt}=\frac{dV}{ds}\frac{ds}{dt}=V\frac{dV}{ds}=\frac{1}{2}\frac{dV^2}{ds}.$$

Since the velocity enters uniformly to the second power in all expressions under consideration we shall make the change of variables $V^2=u$. Before writing the equations of motion we shall make another simplification. As the machine descends it encounters air of increasing density. Therefore $K_y(z)$, $K_x(z)$, $k(z)$ are all increasing functions. We write

$$K_y(z)A=af(s),\quad K_x(z)A+k(z)s=bf(s),$$

where the function $f(s)$ is an increasing function of $s$, assuming that the machine constantly descends. (The fact that $s$ has also been used to represent the detrimental surface will evidently cause no trouble, as it will no longer appear in that significance on account of the substitutions above.) We shall measure arc from the point $P$. We thus have $f(0)=1$.

The equations of motion are therefore:

$$mu\frac{d\theta}{ds}=W\cos\theta-af(s)u,$$

$$\frac{m}{2}\frac{du}{ds}=T+W\sin\theta-bf(s)u,$$

where $m$ represents the mass of the machine.

**33. Rectilinear Descent.**—We shall now investigate the conditions under which rectilinear descent is possible. Since $\theta$ is constant we have from the first equation of motion,

$$af(s)u = W\cos\theta,$$

which gives

$$u = \frac{W\cos\theta}{af(s)}.$$

The second equation then gives as a value of the traction,

$$T = \frac{mW}{2}\cos\theta\,\frac{d}{ds}\left(\frac{1}{af(s)}\right) - W\cos\theta\left(\tan\theta - \frac{b}{a}\right).$$

It is possible to put the quantity $af(s) = K_y(z)A$ in a form more convenient for the present case. We shall write

$$K_y(z) = K_y\rho(z),$$

where $\rho(z)$ denotes the ratio of the density of the air at altitude $z$ to the density at the surface of the earth, and $K_y$ is the value of the lift coefficient at the ground. We then have

$$\frac{d}{ds}\left(\frac{1}{af(s)}\right) = \frac{d}{ds}\left(\frac{1}{AK_y\rho(z)}\right) = \frac{d}{dz}\left(\frac{1}{AK_y\rho(z)}\right)\frac{dz}{ds} = -\frac{\sin\theta}{AK_y}\frac{d}{dz}\left(\frac{1}{\rho(z)}\right).$$

The traction can then be written

$$T = -W\cos\theta\left[\frac{m\sin\theta}{2AK_y}\frac{d}{dz}\left(\frac{1}{\rho(z)}\right) + \tan\theta - \frac{b}{a}\right].$$

Suppose that the angle of descent is given by the relation

$$\tan\theta = \frac{b}{a}.$$

Before making the simplification that results from this assumption we note that $W/K_yA$ is the square of the horizontal velocity at the ground for the given angle of attack. Denote it by $\overline{V}^2$. Then

$$T = -\frac{\overline{V}^2 m\sin 2\theta}{4}\frac{d}{dz}\left(\frac{1}{\rho(z)}\right).$$

If the air were of constant density, we see that it would be possible for a machine to glide in a straight line without any tractive force being supplied. For a given angle of attack

there is a single angle of glide for which this is possible, namely, the angle given above. It is seen to be the angle $\phi$ in the polar diagram. In this case the component of the weight along the path exactly balances the drag, and, if the machine is started with the proper velocity along this course, it will maintain the course without change of velocity.

With the air varying in density the situation is quite otherwise. We have now for the square of the velocity

$$u=\frac{W\cos\theta}{af(s)}=\frac{W\cos\theta}{A\bar{K}_v\rho(z)},$$

which decreases with the altitude. If the angle of descent is that chosen above, the component of the weight along the path balances the drag, and hence an outside force must be used to cause the retardation. The necessary retarding force is of course small, on account of the slow change of the density of the air.

**34.** We have found the value of the traction with the assumption of rectilinear descent. We must reverse the question and assure ourselves that by starting from proper initial conditions rectilinear descent *will* result if the proper traction is constantly furnished.

We start with the equations

$$mu\frac{d\theta}{ds}=W\cos\theta-af(s)u,$$

$$\frac{m}{2}\frac{du}{ds}=T+W\sin\theta-bf(s)u,$$

and suppose that the initial conditions, those for $s=0$, satisfy the relations

$$W\cos\theta-af(s)u=0,$$

$$W\sin\theta-bf(s)u=0,$$

that is, initially,

$$\tan\theta=\frac{b}{a},\quad u=\frac{W\cos\theta}{af(s)}.$$

Put

$$v=\frac{W\cos\theta}{af(s)},$$

and suppose that the traction is given constantly by the expression

$$T=\frac{m}{2}\frac{dv}{ds}.$$

The equations of motion can then be written,

$$mu\frac{d\theta}{ds}=W\cos\theta\left[1-\frac{u}{v}\right],$$

$$\frac{m}{2}\frac{d}{ds}(u-v)=W\cos\theta\left[\tan\theta-\frac{b}{a}\frac{u}{v}\right]$$

$$=W\cos\theta\left[\tan\theta-\frac{b}{a}\left\{1+\frac{u-v}{v}\right\}\right].$$

Making use of the definition of $v$, and putting

$$w=u-v,$$

the equations take the form

$$mu\frac{d\theta}{ds}=-awf(s),$$

$$\frac{m}{2}\frac{dw}{ds}=W\cos\theta\left[\tan\theta-\frac{b}{a}\right]-bwf(s).$$

The quantity $u$ still occurs in the first equation, but will cause no inconvenience. The initial conditions are $\theta=\tan^{-1}\frac{b}{a}$, $w=0$, while $u$ initially has a definite positive value.

Consider an interval $0\leq s\leq s'$ so small that within it $w$ and $\theta$ have no extrema, and therefore their derivatives are not zero except for $s=0$. Therefore throughout the interval $w$ and $\theta$ are monotonic.

Suppose $w$ were increasing, so that $\frac{dw}{ds}>0$. Then since $wf(s)$ is increasing, and both terms on the right-hand side of the second equation are initially zero, we see from that equation that $\theta$ must increase. Hence $\frac{d\theta}{ds}>0$. But this evidently contradicts the first equation.

Suppose next that in the interval considered $w$ is decreasing, and therefore negative, since it is initially zero. The second term on the right of the second equation will be $>0$ in the interval, so that we must have $\tan\theta<\frac{b}{a}$, for $\frac{dw}{ds}$ must be $<0$. Therefore $\theta$ must be decreasing. But the right-hand side of the first equation is now positive. Hence $\frac{d\theta}{ds}>0$. There is therefore again a contradiction.

It follows that throughout the interval we must have $w=0$. Therefore

$$u=v=\frac{W\cos\theta}{af(s)}=\frac{W\cos\theta}{AK_y\rho(z)}.$$

It also follows that

$$\theta=\text{constant}=\tan^{-1}\frac{b}{a}.$$

We see that the initial conditions that we have assumed are maintained throughout the interval $0\leqq s\leqq s'$, and therefore by a process of continuation will be maintained throughout the motion. Therefore rectilinear motion will assuredly result from the initial conditions and the value of the traction that we have assumed.

**35.** We shall next consider the path of descent that is followed when the machine is descending with the propeller not running. The equations of motion are

$$mu\frac{d\theta}{ds}=W\cos\theta-af(s)u,$$

$$m\frac{du}{ds}=2W\sin\theta-2bf(s)u.$$

We have already demonstrated that, if the density of the air remained constant, it would be possible to glide down a straight line, provided the machine started in the correct direction with the proper velocity. In the present problem we shall assume these same initial conditions. That is, for $s=0$, we assume the right-hand sides of the two equations to have the value zero. We recall further that $f(s)$ is an increasing function.

We consider as before an interval $o \leqq s \leqq s'$ within which $u$ and $\theta$ are monotonic. Suppose $u$ increases; then $\frac{du}{ds} > o$, and the second equation shows that $\theta$ must increase, that is, $\frac{d\theta}{ds} > o$. But from the first equation, since $\theta$ increases $\cos\theta$ decreases, and we see that the right-hand side, being zero initially, becomes negative. Hence $\frac{d\theta}{ds} < o$. Therefore there is a contradiction. Hence $u$ cannot increase. The same reasoning shows that $u$ cannot remain constant throughout the interval.

Suppose next that $u$ decreases. Hence $\frac{du}{ds} < o$. The quantity $f(s)u$ may be either increasing or decreasing. As far as the second equation then is concerned, we see that $\theta$ may be increasing or decreasing. Suppose it is increasing. Then $\frac{d\theta}{ds} > o$. In order that $\theta$ may be increasing the second equation shows we must assume $f(s)u$ to be increasing. The first equation then has its right-hand side negative, since $\cos\theta$ decreases, and $f(s)u$ increases. This gives a contradiction, since with $\theta$ increasing we have $\frac{d\theta}{ds} > o$. Suppose, now, that $\theta$ decreases; then $f(s)u$ may be either increasing or decreasing so far as the second equation is concerned. In this instance $\frac{d\theta}{ds} < o$. As $\cos\theta$ increases the first equation shows that $f(s)u$ must be increasing. But no contradiction is reached. Therefore $u$ and $\theta$ both decreasing are the only possibilities. Consequently as the machine descends both $u$ and $\theta$ decrease.

It is apparent that the right-hand sides of the equations having once become negative, $u$ and $\theta$ must continue to decrease. As far as our analysis has shown, the right-hand sides might subsequently have the values zero. But in that instance the preceding development shows that they must then again become negative.

We have thus proved that, if the machine is started down-

wards under conditions that would cause a rectilinear glide if the air density remained constant, the angle of descent becomes continually less, and the velocity likewise diminishes. In the vicinity of the earth the density of the air does not change very rapidly. Therefore the departure from the straight line of the path that a machine follows when gliding is not great, if the descent is from a low altitude. It is, in fact, customary to speak of rectilinear glide as an actual state of possible motion.

**36.** We shall assume that the descent is from a low altitude, so that the course can be regarded as rectilinear. There is now equilibrium between the air reaction and the weight, so that we have the simplified equations of motion,

$$W \cos \theta = K_y A V^2,$$

$$W \sin \theta = K_x A V^2 + ksV^2,$$

where we are using the values of the various coefficients at the surface of the earth.

The angle $\theta$ will be the angle $\phi$ in the polar diagram, as is seen by taking the quotient of the last two equations. It will be the smallest when the angle of attack is the optimum angle, which is generally between 7° and 8°. It is therefore the angle of attack to be used in case it is desired to make the longest possible glide from a given height.

The velocity along the path is given by

$$V = \sqrt{\frac{W}{A} \cdot \frac{1}{K_y}} \cdot \sqrt{\cos \theta}.$$

If we let $\overline{V}$ be the horizontal velocity for the same angle of attack, we have

$$V = \overline{V} \sqrt{\cos \theta},$$

which shows that the velocity along the curve is less than in horizontal flight.

Squaring and adding the equations we find

$$V^2 \sqrt{(K_x + ks/A)^2 + K_y^2} = \frac{W}{A}.$$

Let $r$ represent the distance from the origin to the point on the polar corresponding to the angle of attack. Then we can write

$$V^2 = \frac{W}{A}\,\frac{1}{r}.$$

It is of course necessary to use the true polar for this purpose. While the latter is poorly adapted to show the fineness of the machine, because it is sensibly parallel to the $K_y$-axis, it gives the value of $r$ without any difficulty.

**37.** The rapidity of vertical descent is $V \sin \theta$. We shall denote it by $v$, and shall find the angle of attack for which it is a minimum. For such an angle of attack the machine would require the longest time to reach the earth from a given altitude. An approximation to the angle in question can be easily obtained. We have

$$v = V \sin \theta = \overline{V}\sqrt{\cos \theta} \cdot \sin \theta = \overline{V} \cdot \tan \theta \cdot \cos^{\frac{3}{2}} \theta.$$

For the values of $\theta$ that occur we can replace $\cos \theta$ by unity without much error, and therefore have approximately

$$v = \overline{V} \cdot \tan \theta = \overline{V} \cdot \tan \phi,$$

where $\phi$ is again the angle used in the polar diagram. But $\tan \phi = \frac{T}{W}$, where $T$ is the traction necessary for horizontal flight. Hence, approximately,

$$v = \frac{\overline{V}T}{W} = \frac{P}{W},$$

where $P$ is the power necessary for horizontal flight. It follows that the angle of attack that will make $v$ a minimum is close to the angle of minimum power, that is, the economical angle, which in § 28 was denoted by $\alpha_2$.

**38.** Denote by $\alpha_3$ the angle of attack that renders $v$ a minimum. While it differs little from $\alpha_2$, it is of interest to determine which of the two is actually the larger, and to ascertain if such a relation is invariable for all machines.

We have

$$v^2 = V^2 \sin^2 \theta = \frac{W}{A} \frac{1}{r} \sin^2 \phi.$$

In place of the angle $\phi$ let us use the ordinary polar angle, namely, the angle that the radius vector makes with the $K_x$-axis, and which we denote by $\beta$. Since $\sin \phi = \cos \beta$, we see that $v$ will be a minimum when $r/\cos^2 \beta$ is a maximum. If we knew the equation of the polar curve, the point for minimum $v$ would be obtained by equating the derivative of $r/\cos^2 \beta$ to zero.

We next consider the angle that gives minimum power. In § 28 we have shown that we can write

$$P = hx/y^{3/4},$$

where $h$ is a constant, and $x$ and $y$ are the Cartesian coordinates of a point on the polar, with $Q$ for origin. When transformed to polar coordinates this becomes

$$P = h \frac{\cos \beta}{r^{1/4} \sin^{3/4} \beta}.$$

To find the angle of attack for minimum power, it is then sufficient to find the point on the polar for which

$$\frac{r \sin^3 \beta}{\cos^2 \beta},$$

is a maximum. The derivative of this with regard to $\beta$ can be written

$$\sin^3 \beta \frac{d}{d\beta}\left(\frac{r}{\cos^2 \beta}\right) + 3r \frac{\sin^2 \beta}{\cos^2 \beta}.$$

Return to the point on the polar that corresponds to the angle $\alpha_3$ for minimum vertical velocity. For that point we have

$$\frac{d}{d\beta}\left(\frac{r}{\cos^2 \beta}\right) = 0.$$

Furthermore, this value of $\beta$ gave a maximum to the function $r/\cos^2\beta$. Consequently,

$$\frac{d}{d\beta}\left(\frac{r}{\cos^2\beta}\right)\lesseqgtr 0,$$

according as $\beta$ is greater or less than the value $\beta_3$ corresponding to $\alpha_3$. Now we know that $\alpha_3$ is not much different from $\alpha_2$, which is known to be greater than $\alpha_1$, the optimum angle. Assuming then that $\alpha_3 > \alpha_1$ we see that on the part of the curve

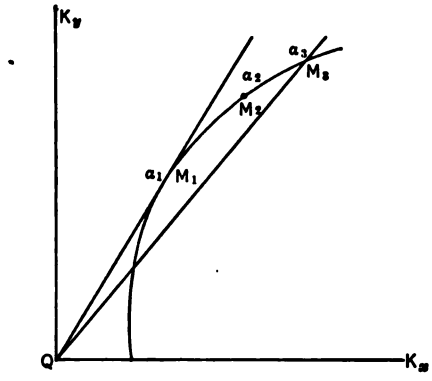


FIG. 19.

corresponding to $\alpha > \alpha_1$ the polar angle $\beta$ is a decreasing function of $\alpha$. Suppose then that $\alpha > \alpha_3$. It follows that $\beta < \beta_3$, and consequently,

$$\frac{d}{d\beta}\left(\frac{r}{\cos^2\beta}\right) > 0.$$

If, however, $\alpha_1 < \alpha < \alpha_3$, then $\beta > \beta_3$, and consequently

$$\frac{d}{d\beta}\left(\frac{r}{\cos^2\beta}\right) < 0.$$

Since all the other quantities that appear in the expressions that we have obtained for the derivative of $r\sin^3\beta/\cos^2\beta$ are positive, we see that that derivative will vanish only if $\alpha < \alpha_3$. Consequently it follows that

$$\alpha_2 < \alpha_3.$$

To make the proof complete suppose that $M_3$, the point corresponding to $\alpha_3$, is to the left of $M_1$, corresponding to $\alpha_1$. Prolong $OM_3$ until it intersects the polar in the second point $M_3'$, for which the corresponding angle of attack is $\alpha_3'$. Then for $\alpha$, any place between $\alpha_3$ and $\alpha_3'$, we would have $\beta > \beta_3$ and consequently

$$\frac{d}{d\beta}\left(\frac{r}{\cos^2\beta}\right) < 0,$$

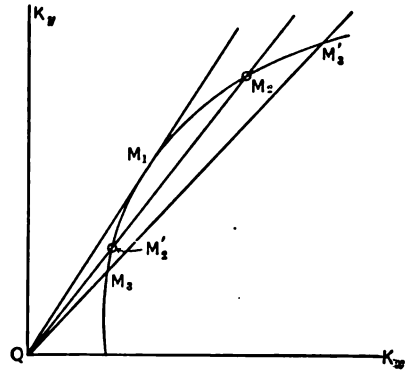


FIG. 20.

and $M_2$ could lie some place between $M_3$ and $M_3'$. It could not lie to the left of $M_3$ or the right of $M_3'$, as in either case $\beta < \beta_3$, and accordingly

$$\frac{d}{d\beta}\left(\frac{r}{\cos^2\theta}\right) > 0.$$

This would give us the opposite of the result obtained by assuming $\alpha_3$ sufficiently near $\alpha_2$ to assure that $M_3$ is to the right of $M_1$. But let us reverse our last reasoning and assume $\alpha_2$ as given. We see at once that $M_3$ must be to the left of $M_2'$, the point used in § 28. It would follow that $\alpha_3$ is not almost equal in value to $\alpha_2$. Therefore it must be that $\alpha_2 < \alpha_3$ is a relation invariable for all machines.

## 2. ASCENT

**39.** Suppose that the machine is in rectilinear horizontal flight, and that a greater admission of gasoline is given to the

motor. The increased traction of the propeller exceeds the air resistance; the velocity of the machine increases, and the lift accordingly exceeds the weight of the machine. The machine starts to rise. The angle of attack changes as a result of the vertical component of the force. The upward motion causes a greatly increased air reaction downwards on the tail plane, and the moment of this force turns the machine in such a way as to keep its axis in the vicinity of the flight path. The complete discussion of the action of the machine must be based upon the considerations given later in the chapter on stability. Oscillations take place, that tend to die out, and there results what can be called a state of steady ascent, with the forces on the machine in equilibrium, and the axis tangent to the trajectory. As the altitude of the machine increases the power necessary for horizontal flight increases. Therefore the excess of power of the motor becomes smaller, and ultimately the machines flies horizontally.

A similar behavior results if the admission of gasoline is not altered, but the angle of attack is changed, by manipulations of the elevator, in such a way as to lessen the power necessary for horizontal flight. The excess power liberated will be absorbed by the rising of the machine.

**40. The Equations of Motion.**—Let $GT$ represent the tangent to the path at any instant, $G$ being the center of gravity of the machine. The forces perpendicular to the tangent are then the lift component of the air reaction and a component of the weight. The forces along the tangent are the drag and a component of the weight, both in a downwards direction, and the traction of the propeller in the opposite direction. Expressing the fact that the forces are in equilibrium, we have the equations

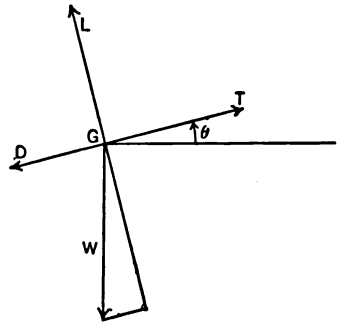


FIG. 21.

$$W \cos \theta = K_y A V^2,$$
$$T = W \sin \theta + K_x A V^2 + ksV^2.$$

**41. The Velocity along the Path.**—If the angle of attack is given, and the direction of the flight path, that is, the angle $\theta$, the first of the equations of motion gives us at once the instantaneous velocity. It is convenient to express this in terms of $\overline{V}$, the velocity for horizontal flight at the same angle of attack. Evidently

$$V = \overline{V} \sqrt{\cos \theta}.$$

It is apparent that as $\theta$ approaches 90° the value of $V$ approaches zero as a limit. The special case $\theta = 90°$ requires extra consideration. In order that the machine should actually mount vertically, like a helicopter, that is, $\theta = 90°$, and $V$ not be zero, it is customary to state that the angle of attack must be that which makes $K_y$ vanish, that is, the angle of zero sustentation. Otherwise there would be a force perpendicular to the flight path. For this angle of attack the velocity $V$ may have any value, provided we furnish sufficient tractive power. But, as noted above, for a *given angle of attack*, other than that of zero sustentation, $V$ approaches zero, when $\theta$ approaches 90°, and we can think of the machine as standing vertically in the air, with any angle of attack, provided the traction exactly balances the weight.

**42. The Force of Traction.**—The traction at any instant is given by

$$T = W \sin \theta + K_x A V^2 + ksV^2.$$

The traction for horizontal flight at the same angle of attack is

$$\overline{T} = (K_x A + ks)\overline{V}^2.$$

Hence,

$$\frac{T - W \sin \theta}{\overline{T}} = \frac{V^2}{\overline{V}^2} = \cos \theta,$$

from which we derive

$$T = W \cdot \sin \theta + \overline{T} \cdot \cos \theta.$$

A simple geometrical construction follows at once. Construct a rectangle of sides $\overline{T}$ and $W$, the latter being vertical. Then $T$ is the sum of the projections of $\overline{T}$ and $W$ on a line making an angle $\theta$ with the horizontal, that is, an angle equal to the inclination of the flight path at that instant. It therefore follows that $T$ is the projection of the diagonal of the rectangle on this line. The diagonal makes with the vertical an angle $\phi$, where $\tan \phi = B$, the fineness of the machine. Since the length of the diagonal is $\overline{T}/\sin \phi$, and since it makes with the line representing the direction of the flight an angle $90° - \phi - \theta$, we can write

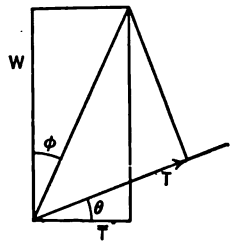

FIG. 22.

$$T = \overline{T}\,\frac{\sin(\theta+\phi)}{\sin \phi} = W\,\frac{\sin(\theta+\phi)}{\cos \phi}.$$

The results that we have derived also show that the traction necessary to mount is the same that it would be if the machine were subject to two forces, its weight and a horizontal force $\overline{T}$.

The traction will be a maximum when the flight path is along the diagonal of the rectangle. In this case $\theta = 90° - \phi$, and

$$T = \frac{\overline{T}}{\sin \phi} = \frac{W}{\cos \phi}.$$

If we take $.15 = \tan \phi$ as a medium value of the fineness, we find for the maximum traction $1.01 \times W$, that is, the maximum traction to be experienced in ascending is 1 per cent greater than the weight of the machine. As long as the angle of ascent is less than $90° - 2\phi$ the traction is less than the weight of the machine. This angle can be as high as 76°, when the fineness is close to .12. As the angle $\theta$ continues to increase from this point the traction exceeds the weight, reaching the maximum given above at a value as high as 83° for the angle of ascent. It then decreases as the angle increases to 90°, on account of the rapidly decreasing speed,

when it again equals the weight, and we have the case of motionless sustentation.

It is obvious that for ordinary angles of ascent the traction will not exceed the weight.

**43. The Power Necessary.**—The power necessary to ascend is $P = T \cdot V$.

On the diagonal of the rectangle used in the last section as diameter construct a circle, omitting that part of the circle that lies below the horizontal and the part to the left of the vertical. On the vertical lay off a distance $\overline{V}$ and to the left of the vertical construct the velocity curve $\overline{V}\sqrt{\cos\theta}$, as shown in Fig. 23. For a given angle of ascent the power is then given by the area of the rectangle $OABC$.

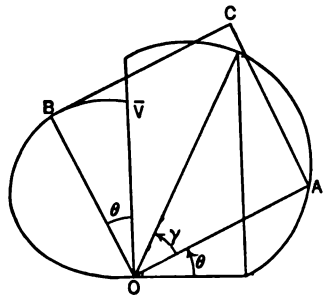


FIG. 23.

Let the diameter of the circle be $D$, and $\gamma$ the angle shown. Then

$$P = D \cos\gamma \cdot \overline{V}\sqrt{\cos\theta}.$$

Hence,

$$\frac{dP}{d\theta} = -D \cdot \overline{V} \sin\gamma \sqrt{\cos\theta}\,\frac{d\gamma}{d\theta} - \frac{D \cdot \overline{V}}{2} \cdot \frac{\cos\gamma}{\sqrt{\cos\theta}} \cdot \sin\theta.$$

Evidently $d\gamma/d\theta = -1$, so that the condition for a maximum (the minimum of $P$ is 0, for $\theta = 90°$, for then $V = 0$) of $P$ is

$$\tan\gamma = \tfrac{1}{2}\tan\theta.$$

For this relation to be satisfied it is evident that $\theta < 90° - \phi$. Therefore, the angle for ascending at which the power is a maximum is less than the angle for which the traction is greatest.

If $\theta_1$ is an angle for which the power necessary to ascend is greater than the power necessary for horizontal flight, there is a second angle $\theta_2$ for which the required power is the same as that for $\theta_1$. The velocity for $\theta_2$ being less than for $\theta_1$, the traction must be greater. (We note, however, that we

cannot say that simply because $\theta_2 > \theta_1$ the traction corresponding to $\theta_2$ must be greater than that corresponding to $\theta_1$, because the traction has been shown to have a maximum.)

**44. The Vertical Velocity.**—The rapidity with which the machine is momentarily moving vertically is

$$v = V \cdot \sin \theta.$$

If the equation of traction is multiplied by $V$ we can write

$$W \cdot v = P - (K_x A + ks) V^3,$$

or in terms of $\overline{V}$,

$$W \cdot v = P - (K_x A + ks) \overline{V}^3 \cos^{3/2} \theta,$$

which can again be simplified into

$$W \cdot v = P - \overline{P} \cos^{3/2} \theta,$$

where $\overline{P}$ is the power necessary for horizontal flight.

If the angle of ascent is small, we can write

$$v = 375 \frac{P - \overline{P}}{W} \text{ miles per hour} = 550 \frac{P - \overline{P}}{W} \text{ feet per second,}$$

where $P$ and $\overline{P}$ are measured in horse-power.

Now $P$ is the useful power that the motor is developing, so that, providing the angle of mounting is small, the velocity of ascent is equal to the excess of power over that required for horizontal flight, divided by the weight. Further, we see that for *a constant available power*, the velocity of ascent is greatest when the angle of attack is $\alpha_1$, that is, the angle for minimum power for horizontal flight. Remembering the results that were obtained for minimum vertical velocity in gliding, we see that an angle of attack that gives a small vertical velocity in gliding, will give a large one in ascending, provided we have a constant available power.

The ideas that have just been adduced must be modified when we come to consider the propelling plant. Changing the angle of attack changes the velocity and traction. This alters the number of revolutions per minute that the propeller must make in order to furnish the traction at that velocity. This in

turn alters not only the efficiency of the propeller, but also the power which the motor is developing. Thus in practice we do not have a constant available power. Consequently the best angle for mounting cannot be expected to be exactly the economical angle.

Another approximation is useful. We have

$$\sin\theta=\frac{v}{V}=\frac{v}{\overline{V}\cos^{\frac{1}{2}}\theta}.$$

We are assuming that practically we can take $\cos\theta=1$, and can therefore take $\theta=\sin\theta$. In degree measure we can therefore write

$$\theta^{\circ}=57\times 375\,\frac{P-\overline{P}}{W\cdot\overline{V}}=22{,}500\,\frac{P-\overline{P}}{W\cdot\overline{V}},$$

approximately, provided the result is a small angle. In this formula the excess useful power is expressed in horse-power, the weight of the machine in pounds, and the velocity in horizontal flight in miles per hour.

**45.** We return to the general considerations of § 39. Suppose the pilot upon leaving the ground gives full admission of gasoline, developing more power than is necessary to fly horizontally near the ground at that angle of attack. The machine starts to rise. The values for the velocity, traction, and power which have been obtained in our discussion are those that exist at any moment. They are known if the direction of the tangent to the path is known. The general problem that arises is of great difficulty. The power of the motor for a specified number of revolutions per minute decreases as the machine gains altitude, because the diminishing atmospheric pressure decreases the mass of gas mixture that is in the cylinder at each explosion. Further, the changing density of the air modifies the propeller's action, and the changing velocity alters the number of revolutions per minute. If the angle of attack is not changed, the curvature of the path decreases with the altitude, and the machine ultimately flies horizontally. This occurs when the power the motor is developing multiplied by the efficiency of the propeller is exactly the power required

for horizontal flight at that altitude. The height to which the machine has risen is called the **ceiling**. It is dependent upon the angle of attack. But the word can be used, without danger of confusion, to denote the height to which the machine can rise for a given angle of attack, or the *maximum of these heights, that is, the greatest height at which the machine can possibly fly.*

As the machine is fitted with various instruments for measuring height, velocity, time, etc., a record of the flight can be made, and from such data experimental laws can be adduced. Though some of these laws may appear only approximate, they will nevertheless allow us to obtain, by equations derivable from them, other results of sufficient accuracy to be both of interest and value. And it is only in this way that certain problems can be attacked.

**46. Experimental Law of Vertical Velocity.**—Experience shows that the vertical velocity decreases sensibly as a linear function of the altitude. Suppose the machine is flying at the angle of attack that allows it to reach its ceiling. Let $h$ be the height of the ceiling, and $v_0$ the vertical velocity at the ground. Then

$$v = v_0\left(1 - \frac{z}{h}\right),$$

where $z$ is the altitude. It is convenient to measure the velocities in feet per second and $h$ and $z$ in feet.

Experience also shows that the initial velocity $v_0$ can be calculated very approximately from a knowledge of the power of the motor, the weight of the machine, and the height of the ceiling. Let $H_0$ be the power of the motor at the earth, using full admission of gas. Then the quantity

$$\pi = \frac{W}{H_0}$$

is the weight per unit horse-power, a quantity, like the loading, of great importance. The quantity $v_0$ is then given approximately by the relation

$$v_0 = \frac{h}{95\pi}.$$

For example, a machine weighing 7.59 pounds per horsepower, and having a ceiling at 24,500 feet, will have an initial vertical velocity of 34 ft./sec.

We can then write

$$v=\frac{h-z}{95\pi}.$$

**47. Time of Ascent.**—We have

$$\frac{dz}{dt}=v_0\left(1-\frac{z}{h}\right).$$

Integrating this we find

$$t(z)=\frac{h}{v_0}\cdot\log_e\frac{1}{\left(1-\frac{z}{h}\right)}.$$

In terms of common logarithms this becomes

$$t(z)=2.303\frac{h}{v_0}\cdot\log\frac{1}{\left(1-\frac{z}{h}\right)},$$

the time $t(z)$ being expressed in seconds.

**48. Determination of the Ceiling.**—The result which has been derived for the time of ascent can be used in a very simple way to determine the height of the ceiling.

We have

$$\left(1-\frac{z}{h}\right)=e^{-t(z)\frac{v_0}{h}}.$$

Let $t(z_1)$ and $t(z_2)$ be the times required to reach altitudes $z_1$ and $z_2$, respectively, $z_1<z_2$, and put

$$\frac{t(z_2)}{t(z_1)}=n.$$

Then

$$\left(1-\frac{z_2}{h}\right)=e^{-t(z_2)\frac{v_0}{h}}=e^{-t(z_1)\frac{v_0}{h}\cdot n}.$$

Hence,

$$\left(1-\frac{z_2}{h}\right)=\left(1-\frac{z_1}{h}\right)^n.$$

Suppose now that in ascending, the pilot notes the time $t(z_1)$ when he reaches any chosen altitude $z_1$, and that he then notes his altitude $z_2$ when the interval $2t(z_1)$ since leaving the ground has elapsed. We have then $n=2$.

Hence

$$\left(1-\frac{z_2}{h}\right)=\left(1-\frac{z_1}{h}\right)^2.$$

When this is expanded and solved for $h$ we find

$$h=\frac{{z_1}^2}{2z_1-z_2}.$$

It is therefore unnecessary for the pilot to determine the ceiling by flying until his altimeter indicates that he is no longer gaining altitude.

Suppose that a machine requires 8 minutes 40 seconds to reach an elevation of 9850 feet, and that in double the interval, that is, in 17 minutes 20 seconds it has reached an altitude of 15,500 feet. We then have for the height of the ceiling,

$$h=\frac{(9850)^2}{2\times 9850-15{,}500}=23{,}100 \text{ feet.}$$

# CHAPTER IV

## CIRCULAR FLIGHT

### 1. Horizontal Turns

**49.** When we consider the question of controllability of an airplane we are at once confronted by the problem of turning. A machine must be equipped with controls which will allow its course to be altered at the will of the pilot, with ease, certainty, and safety. It is found at once that the successful solution of the problem gives to the airplane properties that distinguish it from other vehicles in the matter of turning.

The airplane will not change its direction from a given vertical plane unless one of the three forces acting on it has a component perpendicular to that plane. Obviously the force that can give this component is the air pressure. Thus the solution of the problem will be obtained by making the air resistance have a component in the direction towards which it is desired to turn. It is thus apparent that the plane of symmetry must be inclined to the vertical, that is, the machine must be banked. The center of gravity will then be urged in the direction towards which the plane of symmetry has been tilted, for the air reaction being in this plane, will now have a component in that direction. If at the same time the proper turning moment can be given the machine, it can be made to keep its axis tangent to the curve described by the center of gravity, and this evidently is necessary for a proper turn. Therefore to effect a turn we must create: (1) a centripetal force, (2) a turning moment.

It will be found that new phenomena arise when we have deflected the machine from a horizontal straight line flight. The machine may slip sideways, that is, in a direction per-

pendicular to the plane of symmetry. This is undesirable. A turn is regarded as correct when the instantaneous direction of motion remains always in the plane of symmetry. There is then no slipping. This can be accomplished, and a machine can be gotten into such a position that, with controls fixed, it will describe a horizontal circular path.

**50. Equations of Motion.**—Suppose an airplane with center of gravity at $G$ is flying in a circle of radius $r$ with center in the direction $GC$. Let $\theta$ be the angle of bank. The lift, being

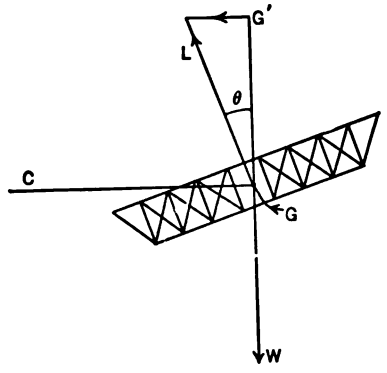


FIG. 24.

in the plane of symmetry, is represented by $GL$, and from it we must obtain both the sustaining force and the turning force. Let $V$ be the velocity with which the machine is moving. We know that in order to make it describe the circular path there must be a force toward the center equal to $mV^2/r$, where $m$ is the mass of the machine.

We therefore have the three equations of motion:

$$W = K_y A V^2 \cos \theta,$$

$$\frac{mV^2}{r} = K_y A V^2 \sin \theta,$$

$$T = K_x A V^2 + k_s V^2.$$

If these equations are satisfied, the center of gravity will momentarily be describing the circle in question. But in order to make the motion similar to that of a rigid body rotat-

ing about an axis there must be a turning moment applied to the machine in order to keep the axis of the machine tangent to the path. The method of producing this moment will be considered later. Furthermore, it is known from the principles of rigid dynamics that when a rigid body rotates around an axis the centrifugal reaction also gives rise to a couple tending to displace the axis, unless the axis is a principal axis of inertia. This couple would manifest itself by an alteration of the angle $\theta$. It must be combated by the controls. The gyroscopic action of the propeller and engine also come into play. They also will be disregarded.

**51. The Velocity and Inclination.**—From the first equation of motion we obtain

$$r = \frac{\bar{V}}{\sqrt{\cos \theta}},$$

where $\bar{V}$ is the velocity for rectilinear flight at the same angle of attack. It therefore follows that the velocity in a circle is greater than that in a straight line. In turning, an airplane must therefore increase its speed, contrary to the practice of other vehicles.

From the first two equations we have

$$\tan \theta = \frac{mV^2}{Wr} = \frac{V^2}{rg},$$

where $g$ is the acceleration of gravity. In terms of $\bar{V}$ we find

$$\tan \theta = \frac{\bar{V}^2}{rg \cos \theta}.$$

Hence,

$$\sin \theta = \frac{\bar{V}^2}{rg}.$$

For a given radius of turn there is therefore only one inclination that will produce a correct turn.

Using this value for $\sin \theta$ we obtain for the velocity the expression

$$V = \bar{V}\left(1 - \frac{\bar{V}^4}{r^2 g^2}\right)^{-\frac{1}{4}}.$$

We therefore have both the inclination and the velocity expressed in terms of $\overline{V}$ and the radius of the turn. While we can to a certain extent regard the radius as an arbitrary element in the turn, we find at once that it is limited. The value for $\sin\theta$ must be less than unity. This shows that we must have

$$r > \frac{\overline{V}^2}{g} = \frac{W}{g}\cdot\frac{1}{AK_y} = \frac{m}{AK_y}.$$

This is consequently an inferior limit to the radius of the turn that a machine can make. It depends upon the angle of attack. It is smallest for the angle of attack that gives the maximum value to $K_y$. It is obvious that as the radius of the turn approaches its inferior limiting value the angle $\theta$ approaches 90° and the velocity $V$ approaches infinity.

In practice, however, a machine has an inferior limit to the radius of a turn it can execute that is much greater than the theoretical one which has been derived. For long before the theoretical limit has been reached, the power of the engine will have become insufficient to furnish the necessary velocity.

**52. The Traction and Power.**—The traction and power are easily expressible in terms of their values $\overline{T}$ and $\overline{P}$ for horizontal flight at the same angle of attack.

We have

$$\frac{T}{\overline{T}} = \frac{V^2}{\overline{V}^2} = \frac{1}{\cos\theta}.$$

Hence

$$T = \overline{T}\left\{1 - \frac{\overline{V}^4}{r^2g^2}\right\}^{-\frac{1}{2}}.$$

In the same way

$$\frac{P}{\overline{P}} = \frac{TV}{\overline{T}\,\overline{V}} = \frac{1}{\cos^{\frac{3}{2}}\theta},$$

so that

$$P = \overline{P}\left\{1 - \frac{\overline{V}^4}{r^2g^2}\right\}$$

Expanding the quantity on the right we have

$$P=\overline{P}\left\{1+\frac{3}{4}\frac{\overline{V}^4}{r^2g^2}+\frac{21}{32}\frac{\overline{V}^8}{r^4g^4}+\ldots\right\}.$$

Hence

$$\frac{P-\overline{P}}{\overline{P}}=\frac{3}{4}\frac{\overline{V}^4}{r^2g^2}\left\{1+\ldots\right\}.$$

Suppose the radius of the turn is large. Then at the same angle of attack the increased power necessary varies approximately as the inverse of the square of the radius.

**53.** The results that have been derived have merely shown how the correct turning force can be obtained by giving the proper inclination to the plane of symmetry. It is necessary to see how the machine can be gotten from its horizontal straight line path into the required banked position, and how the tendencies to depart from this position can be overcome.

The controls that are used for the purpose are the rudder and the ailerons.* Suppose the rudder is turned towards the right.† A moment is produced turning the machine in that direction, and at the same time a small force tending to make the center of gravity move slightly to the left. The ultimate effect of the air pressure on the rudder and the keel surface is to start the machine turning towards the right. This causes the left part of the machine to move faster than the right. Consequently, the air pressure on the left is greater than on the right, and the machine starts to bank towards the right, which is the correct direction to produce a turn to the right. This banking is not equal in general to that required for a correct turn. The pilot increases the degree of bank by the ailerons. The ailerons on the left are depressed and those on the right are raised, thus producing a greater lift on the left wing. The machine having been banked, the lift on the wing, which before the turn was vertical and just equal to the weight, now departs from the vertical, so that the component that acts in the direction opposite to gravity is no longer sufficient to

* For a description of the ailerons see § 98.

† See §§ 99, 100 for a fuller explanation of the rudder and its action.

sustain the machine, and the machine will start to dive, unless its velocity is increased. For this purpose the pilot may either increase the admission of gas, or alter his angle of attack, so as to obtain an excess of power. If it should happen that his machine is "tangent," that is, an excess of power cannot be secured in either of these ways, he cannot turn, without descending to a lower altitude, and thus liberating some excess of power.

As the outside of the machine moves faster, the drag there will be greater than on the side towards the center of the turn. This has a tendency to make the machine turn towards the left, and it must be overcome by the controls. The turning moment on the machine necessary to keep the axis along the path is secured from the rudder.

We have seen that in order to turn correctly it is necessary to produce a turning moment and a centripetal force. In general, we can say that the function of the rudder is to produce the necessary turning of the axis of the machine, and the function of the ailerons to give the banking that is required to produce the necessary centripetal force.

It is also by means of the ailerons that a tendency to slip is overcome. If the machine tends to slip towards the outside of the curve the ailerons on that side are depressed and those on the other side raised. This banks the machine further towards the inside, throwing the resultant air force further from the vertical, and thus increases the force towards the center. In case the machine starts to slip towards the inside of the curve the ailerons are manipulated in the contrary way.

When it is desired to straighten a machine out after a change of direction has been accomplished it is necessary to make operations the reverse of those described for making the turn.

When making a turn the pilot must guard against heading upward, because the loss of speed may be too great to insure stability, unless a large excess of power is available.

It is evident that in a banked position the function of the elevator and the rudder are not distinct, but have closely related effects, depending upon the degree of inclination.

In some early machines, turning was effected without banking the machine. The necessary centripetal force on the center of gravity was secured by the air pressure on vertical partitions. When the rudder was in a neutral position, there was no pressure on these surfaces, but when the rudder was turned, a force arose that produced the turn. A good deal of slipping arose in such a turn, and therefore the turn could not be considered as correct. Furthermore, the drag on the vertical surfaces reduced the speed of the machine.

## 2. Circular Descent

**54.** We shall next consider the question of helical descent, without the motor running. The traction necessary to overcome the drag of the machine will be furnished by the weight of the machine, and the necessary centripetal acceleration will be derived from the air reaction, the machine being again in a banked position. We shall assume that the air density can be considered constant in the distance through which we are considering the motion.

**55.** Let $AGB$ represent the path of the machine, $G$ being the position of the center of gravity at any instant. Let $r$ represent the radius of the cylinder on which we are supposing the helical path is traced. Let $CGD$ be the horizontal circle through $G$. The weight of the machine is represented by $GW=W$, drawn vertical. The centrifugal force is represented by $GH=H$, outward along the radius. The resultant of these two forces is represented by $GQ=Q$, in the plane through the axis of the cylinder and point $G$, and making an angle $\theta$ with the vertical. The air reaction must then be represented by $GR$, opposite in direction and equal numerically to $GH$. The velocity of the machine is represented by $GV=V$, the horizontal component by $GV_0=V_0$. Let further $\psi$ be the angle between $GV$ and $GV_0$.

In order that the turn may be properly executed it is necessary that the axis of the machine coincide instantaneosuly with the tangent to the path. Therefore the plane of sym-

metry is determined by $GR$ and $GV$. The lift component of the air reaction lies in this plane, being perpendicular to $GV$, and the drag is along $GV$. The resultant $Q$ of $W$ and $H$ also lies in the plane of symmetry. We can then consider that the force $Q$, the lift $L$, and the drag $D$ are in equilibrium. In the

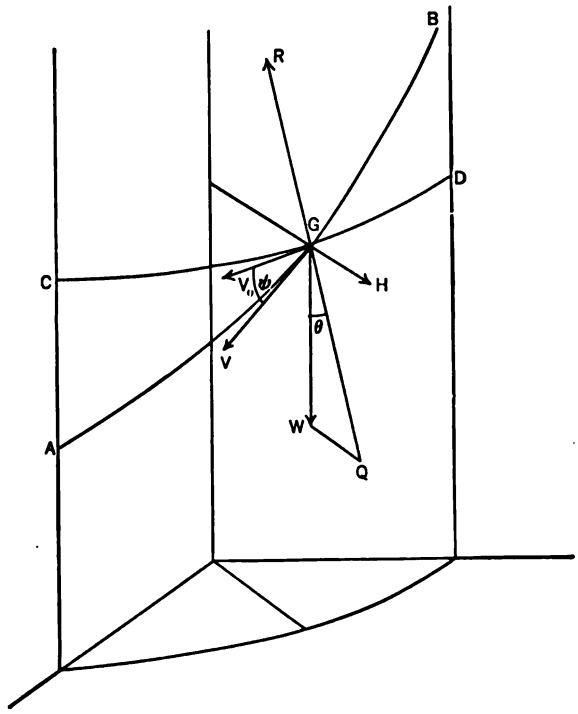


FIG. 25.

plane of symmetry draw a line perpendicular to $GQ$. Let the angle between it and $GV$ be $\psi'$. Since the lift $L$ is perpendicular to $GV$, we have, upon resolving forces perpendicular to and along $GV$, the following equations:

$$Q \cos \psi' = K_y A V^2,$$

$$Q \sin \psi' = K_x A V^2 + ksV^2,$$

as the equations of sustentation and traction, respectively. The angle $\theta$ is evidently given by the relation

$$\tan\theta=\frac{mV_0^2}{Wr}=\frac{V_0^2}{gr},$$

where $m$ is the mass of the machine.*

**56.** The angles $\theta$, $\psi$ and $\psi'$ are evidently not independent, and the relation between them can be easily found by spherical trigonometry. Let $GZ$ represent the vertical through $G$, $GX$

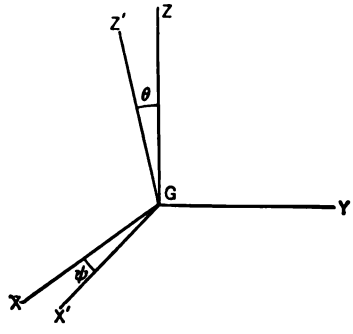


FIG. 26.

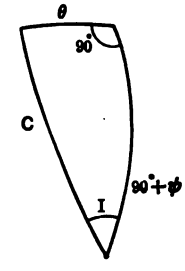


FIG. 27.

the tangent to the circular section $CGD$, and $GY$ the radius continued outward. Draw $GZ'$ in the plane of $YZ$, making an angle $\theta$ with $GZ$. Likewise draw $GX'$ in the plane of $XZ$, making an angle $\psi$ with $GX$. Then evidently the plane of $GZ'$ and $GX'$ represents the plane of symmetry of the machine. About $G$ describe a sphere of unit radius. We thus have a right spherical triangle which we can represent separately from the axes, as shown in the Fig. 27. We have then

$$\cos C=\cos\theta\cdot\cos(90^\circ+\psi).$$

If we recall the definition of $\psi'$, we see that it is the angle between $GX'$ and a line in the plane of $GZ'$ and $GX'$ and perpendicular to $GZ'$. It then follows that

$$C=90^\circ+\psi'.$$

* Devillers, "La Dynamique de l'Avion," p. 135.

The preceding relation then becomes

$$\sin \psi' = \cos \theta \sin \psi,$$

from which we note in passing that

$$\psi > \psi'.$$

We can also find the inclination of the plane of symmetry. This inclination is the angle between the plane of symmetry and the vertical, that is, the angle $I$ in the right spherical triangle we have constructed. Therefore,

$$\tan I = \frac{\tan \theta}{\cos \psi}.$$

This relation can be put in another form which will later be useful. We have

$$\cos I = \frac{\cos \psi}{\sqrt{\cos^2 \psi + \tan^2 \theta}} = \frac{\cos \psi}{\sqrt{\sec^2 \theta - \sin^2 \psi}}.$$

$$= \frac{\cos \psi \cos \theta}{\sqrt{1 - \sin^2 \psi \cos^2 \theta}}.$$

Therefore,

$$\cos I = \frac{\cos \psi \cdot \cos \theta}{\cos \psi'}.$$

Hence,

$$\frac{\cos \psi'}{\cos \theta} = \frac{\cos \psi}{\cos I}.$$

**57. Other Form of Equations of Motion.**—We can put the equations of motion in another form. The path described is a skew curve, whose osculating plane contains the tangent and the radius of the cylinder. The principal normal is therefore along the radius; the binormal is perpendicular to the tangent and the principal normal.

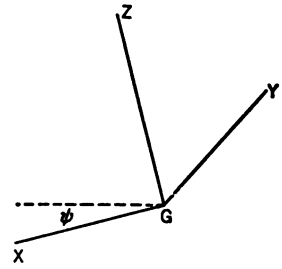


FIG. 28.

We shall resolve the forces along the tangent, principal normal and binormal, represented, respectively, by $GX$, $GY$, and $GZ$.

As the weight of the machine acts vertically downwards, its components are respectively,

$$W \sin \psi, \qquad 0, \qquad -W \cos \psi.$$

Consider next the air reaction. It acts in the plane of symmetry, which passes through $GX$ and makes an angle $I$ with the vertical plane $XGZ$. The lift $L$ lies in this plane, and is perpendicular to $GX$. Therefore its components are

$$0, \qquad L \sin I. \qquad L \cos I.$$

The drag is along $OX$, and its components are

$$-D, \qquad 0, \qquad 0.$$

Finally, we have to consider the centrifugal force, $mV_0^2/r$, where $V_0$ is the horizontal projection of the velocity. This force is along the radius, so its components are

$$0, \qquad -\frac{mV_0^2}{r}, \qquad 0.$$

For steady flight the sum of the forces along each axis must be zero. Hence we have:

$$D = W \sin \psi,$$

$$L \sin I = \frac{mV_0^2}{r},$$

$$L \cos I = W \cos \psi.$$

If we divide the second equation by the third, we can take, as the equations of motion: *

$$D = W \sin \psi,$$

$$L \cos I = W \cos \psi,$$

$$\tan I = \frac{V_0^2}{rg \cdot \cos \psi}.$$

**58.** We next show that the equations of motion last derived are identical with the first. In the latter we replace $Q$ by $W/\cos \theta$, and insert $L$ and $D$ on the right-hand sides of the equations of sustentation and traction, respectively. The

* Cowley and Evans, loc. cit., p. 222.

former equations then become, when written in the order of those in the last paragraph:

$$D=\frac{W}{\cos\theta}\sin\psi',$$

$$L=\frac{W}{\cos\theta}\cos\psi',$$

$$\tan\theta=\frac{V_0^2}{gr}.$$

If now we make use of the relations:

$$\sin\psi'=\cos\theta\sin\psi,$$

$$\frac{\cos\psi'}{\cos\theta}=\frac{\cos\psi}{\cos I},$$

$$\tan\theta=\tan I\cdot\cos\psi,$$

which were derived in § 56, we see that the equations of motion last written become identical with those obtained in the last paragraph.

**59.** From the equations of sustentation and traction we have

$$\tan\psi'=\frac{K_x+ks/A}{K_y},$$

so that $\psi'$ is equal to the angle of glide for rectilinear descent, and is therefore uniquely determined by the angle of attack.

We shall again regard $r$ as an independent variable, specifying the conditions of flight, as in the case of circular flight, and determine the other quantities that specify the motion, namely, $\psi$ and $\theta$ in terms of it and $\psi'$, which we have just seen is determined by the angle of attack.

When we substitute

$$V_0=V\cos\psi,$$

in the third equation of motion we obtain

$$\tan\theta=\frac{V^2\cos^2\psi}{rg}.$$

Making use of the equation of sustentation this becomes

$$\tan\theta=\frac{Q\cos\psi'\cos^2\psi}{rgK_yA}.$$

But

$$Q=\frac{W}{\cos\theta'}$$

so that we have finally

$$\sin\theta=\frac{m}{rK_yA}\cos\psi'\cos^2\psi,$$

$$=\frac{m}{rK_yA}\cos\psi'\left[1-\frac{\sin^2\psi'}{\cos^2\theta}\right],$$

upon making use of the relation between $\psi$, $\psi'$ and $\theta$.

When we replace $\sin^2\psi'$ by $1-\cos^2\psi'$, and $\cos^2\theta$ by $1-\sin^2\theta$ the last relation reduces to

$$\sin^3\theta-\frac{m\cos\psi'}{rK_yA}\sin^2\theta-\sin\theta+\frac{m}{rK_yA}\cos^3\psi'=0,$$

a cubic equation for $\sin\theta$, in which all the coefficients are known as soon as we know the radius of the helix and the angle of attack.*

Put

$$x=\sin\theta, \qquad a=\frac{m\cos\psi'}{rK_yA}.$$

The equation then reduces to

$$x^3-ax^2-x+a\cos^2\psi'=0,$$

where $x$ is the unknown, and $a$ and $\psi'$ are known. This equation can be written

$$(x-a)(x^2-1)=a\sin^2\psi',$$

or

$$\left(1-\frac{x}{a}\right)=\frac{\sin^2\psi'}{1-x^2}.$$

We construct now the curve,

$$y=\frac{\sin^2\psi'}{1-x^2},$$

* The cubic equation for $\sin\theta$ is given by Devillers, loc. cit., p. 138. His discussion of the equation is quite different from that given here.

which consists of the three branches shown, asymptotic to the lines $x+1=0$, $x-1=0$. Draw finally the line

$$y=1-\frac{x}{a},$$

which passes through (0,1) and has a slope of $-1/a$.

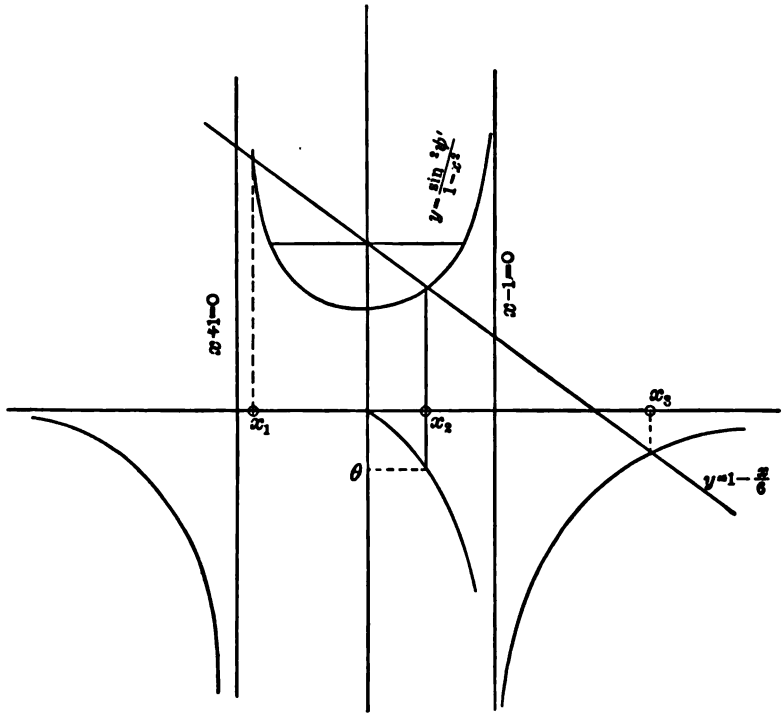


FIG. 29.

The roots of the cubic then correspond to the abscissæ of the points of intersection of the line and the curve. It is obvious that all three of the roots are real. Let them be denoted by $x_1$, $x_2$, and $x_3$. It is seen that

$$-\infty < x_1 < 0 < x_2 < 1 < x_3 < \infty .$$

Since $x=\sin\theta$, the only root which is applicable is $x_2$. We thus have $\sin\theta$, and consequently $\theta$, uniquely determined.

The diagram can be easily modified so as to allow the value of $\theta$ to be directly obtained. Draw the curve $x=\sin\theta$, taking for $\theta$ the negative $y$-axis, as shown. By continuing the ordinate through $x=x_2$ until it intersects the curve $x=\sin\theta$, the value of $\theta$ can be immediately obtained.

It is interesting to consider the limiting values for $\theta$ when $r$ approaches zero and infinity, respectively. The corresponding values of $a$ are evidently $\infty$ and 0. The slope of the line we have used becomes $-\infty$ for $a=0$, and evidently the corresponding value of $x_2$ is zero. Thus for $r=\infty$ we have $\theta=0$, and the descent becomes rectilinear.

Let $r=0$, so that $a=\infty$. The root $x_2$ is then determined by

$$1=\frac{\sin^2\psi'}{1-x^2_2}.$$

This gives

$$\sin^2\theta=1-\sin^2\psi'=\cos^2\psi',$$

and therefore

$$\theta=90^\circ-\psi'.$$

This is a maximum value for the angle $\theta$.

After we have obtained the values of $\psi'$ and $\theta$, we can at once get $\psi$, the angle of descent, from the relation

$$\sin\psi=\frac{\sin\psi'}{\cos\theta}.$$

In the limiting case $r=0$ we have $\sin\psi=1$, so that $\psi=90^\circ$. The machine in this case descends vertically, rotating about its axis as it descends. The value of $I$ is seen to be equal to 90°.

**60. The Velocity.**—We have from the first two equations of § 55

$$Q=[(K_xA+ks)^2+K_y^2A^2]^{\frac{1}{2}}V^2.$$

Hence,

$$\frac{W}{A}\cdot\frac{1}{\cos\theta}=V^2\sqrt{(K_x+ks/A)^2+K_y^2}.$$

Let $V'$ be the velocity for a rectilinear descent at the same angle of attack. From § 36 it follows that

$$V^2 = \frac{V'^2}{\cos\theta},$$

and therefore $V$ can be determined as soon as $\theta$ has been found. We see that the velocity in a helical descent is greater than in a rectilinear glide. In the limit for $r=0$, we have $\cos\theta = \sin\psi'$. Hence the limit of $V$ is given by

$$V^2 = \frac{V'^2}{\sin\psi'} = V'^2\sqrt{1+\frac{1}{B^2}},$$

where $B$ is the fineness.

The distance that the machine will advance downwards for each revolution about the axis of the cylinder, that is, the pitch of the helix, is

$$h = 2\pi r \cdot \tan\psi.$$

In the limit as $r$ approaches zero, this becomes indeterminate, for $\psi$ then approaches 90°. In order to investigate the limit we shall express both $r$ and $\psi$ in terms of $\theta$ and $\psi'$. We have when this is done:

$$\tan\psi = \frac{\sin\psi'}{\sqrt{\cos^2\theta - \sin^2\psi'}} = \frac{\sin\psi'}{\sqrt{\cos^2\psi' - \sin^2\theta}}.$$

From § 59 we find,

$$r = \frac{m\cos\psi'}{K_y A}\cdot\frac{\cos^2\psi' - \sin^2\theta}{\sin\theta\cdot\cos^2\theta}.$$

Hence,

$$h = \frac{\pi m}{K_y A}\cdot\frac{\sin 2\psi'\sqrt{\cos^2\psi' - \sin^2\theta}}{\sin\theta\cdot\cos^2\theta}.$$

In the limit when $r=0$ we have $\sin\theta = \cos\psi'$. Hence in the limit $h=0$.

If we write,

$$\frac{h}{r} = 2\pi\cdot\tan\psi,$$

and note that when $r$ approaches infinity the angle $\psi$ approaches $\psi'$, we see that the graph of $h$ as a function of $r$ approaches the line

$$h = r \cdot 2\pi \tan \psi' = 2\pi B \cdot r$$

asymptotically.* But we know that $\psi' < \psi$, so that the graph of $h$ as a function of $r$ would be as shown in Fig. 30.

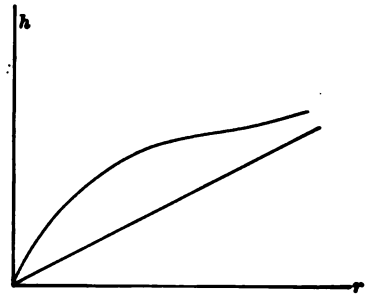


FIG. 30.

Suppose the machine is descending at an angle of attack such that the fineness is .15, and that the radius of the helix is 300 feet. Then we have for the minimum the machine can advance for each revolution

$$h = 300 \times 2\pi \times .15 = 282 \text{ feet.}$$

* Devillers, loc. cit., p. 148.

## CHAPTER V

### THE PROPELLER

**61.** The power of the motor with which an airplane is equipped is transformed into thrust by means of a propeller. We can, in fact, define, in a general way, a propeller as an instrument which will furnish, by the air reaction upon it, a thrust along an axis, when it is rotated about that axis. Considerable variation in design exists in propellers that are used at the present time, and the entire field is one in which continued investigation is necessary. There are, however, basic principles which are followed in propeller design, and general laws that determine their behavior. In the short treatment that is given here only the more general aspects of the problem are considered.

**62.** The study of a propeller's action is one of great difficulty. It involves both theoretical and experimental considerations. It is easy to see that a complete solution of the problem by means of mathematical processes is hardly to be hoped for. With the speed with which a propeller rotates the problem becomes much more complex than that of the action of an aërofoil. The compressibility of the air will play an important part. The air in front of the propeller is drawn in towards it, so that the hypothesis that the propeller acts upon air at rest will be only very approximate. Nevertheless theoretical considerations are very important. They can give us at least tentative laws that we can subject to tests, and they help us to discover experimental laws, by suggesting what may be the relation between various observed phenomena.

**63.** The laws that we have applied to the consideration of the action of an aërofoil furnish a basis for the design of a

propeller, and will be used in the following consideration of a propeller's action.

Let Fig. 31*a* represent one blade of a propeller. Consider a section $AB$. The shape of the section is shown in Fig. 31*b*. It is seen that the section resembles that of an aërofoil. As in aërofoils, variation exists in the amount of camber of the

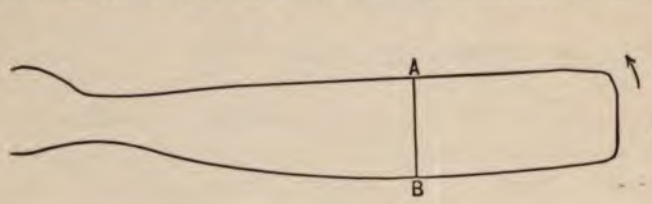


FIG. 31*a*.

FIG. 31*b*.

upper and lower surfaces. The general plan of the blade also varies.*

**64.** Consider the air reaction on a section of the propeller at distance $r$ from the axis. Suppose the propeller is making $n$ turns a second, and advancing along the axis at the rate $V$ feet a second. At a given instant the section has therefore two velocities, namely, a velocity $u = 2\pi nr$ in a direction normal to the axis, and the velocity $V$ along the axis. The resultant $V'$ of these two velocities is represented by $AC$. We consider that the section is instantaneously moving in the latter direction. The angle of attack is therefore the angle $\phi$. The resultant air reaction $R$ will be in a direction near to the normal to the chord of the section. It will have two components,

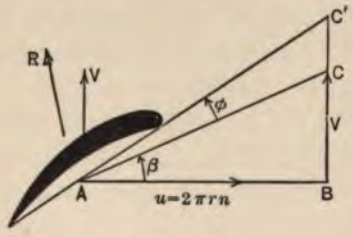


FIG. 32.

* For a discussion of various designs in propellers, see the report by W. F. Durand in the Third Annual Report of the National Advisory Committee on Aeronautics, 1917, Report No. 14, entitled "Tests on 48 Model Forms of Air Propellers with Analysis and Discussion of Results and Presentation of the same in Graphic Form."

one in a direction opposite to the rotation, and the other along the axis. The first component opposes the motor couple, and the second produces a thrust along the axis. A similar analysis holds for each element of the blade, and we see that the resultant effect of the whole propeller is the production of a couple that must be overcome by the motor, and a thrust along the axis of the rotation.

**65. Geometrical Pitch.**—In the figure let $\theta = \beta + \phi$. The value of $\theta$ depends only on the inclination of the section, which we shall call the **setting** of the section. A consideration of the figure leads us to regard the chord of the section as a part of a geometrical helix of radius $r$, that makes an angle $\theta$ with a section normal to the axis of the helix. From this viewpoint one revolution should make the element advance a distance

$$H = 2\pi r \cdot \tan \theta.$$

This is called the **geometrical pitch** of the element. It will vary from element to element, unless the angle $\theta$ for an element is connected with the radius of the element by the relation

$$\theta = \tan^{-1} \frac{H}{2\pi r},$$

where $H$ is some constant.

There are then two classes of propellers: those of constant pitch, and those of variable pitch.

(1) *Propellers of Constant Pitch.*—By choosing the angle $\theta$ to satisfy the relation given above the pitch of the propeller will be made constant. The angles at which various elements of the blade are set are easily shown graphically. We construct the distance $H/2\pi$ as an ordinate, giving the point $P$, and on the axis of abscissæ lay off distances equal to different fractions of the propeller's radius. The corresponding points are joined to $P$. The lines obtained evidently give the inclination of the chords of the various sections to a plane normal to the axis of the propeller.

Suppose the propeller is of constant pitch, and that it

advances due to the thrust it creates at the velocity $V$. We have then for the angle $\beta$, Fig. 32,

$$\beta = \tan^{-1} \frac{V}{2\pi nr},$$

so that the angle of attack is

$$\phi = \theta - \beta = \tan^{-1} \frac{H}{2\pi r} - \tan^{-1} \frac{V}{2\pi nr}$$

$$= \tan^{-1} \frac{\frac{H}{2\pi r} - \frac{V}{2\pi nr}}{1 + \frac{H \cdot V}{(2\pi r)^2 \cdot n}} = \tan^{-1} \frac{nH - V}{(2\pi r)^2 n + HV}.$$

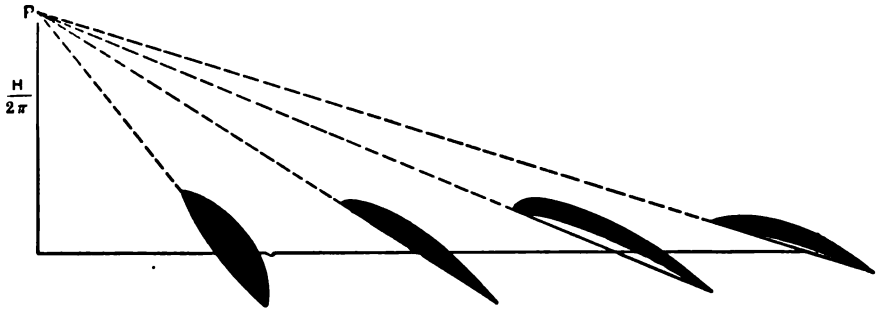


FIG. 33.

This depends upon $r$, so the angle of attack is different for different elements of the blade, being the smallest at the tip, and increasing towards the axis. This fact can be considered an objection to a propeller of constant pitch, although the ultimate decision must depend on experiment.

(2) *Propellers of Variable Pitch.*—The angle $\theta$ can be chosen so that the angle of attack of all elements will be the same, by merely determining $\theta$ by the relation

$$\theta = \phi + \tan^{-1} \frac{V}{2\pi nr},$$

where $\phi$ is constant. The manner of graphically obtaining the inclination of the various sections is shown in Fig. 34. The

ordinate is constructed of length $V/2\pi n$. Lines are drawn to various points on the axis of abscissæ as in the former case. An angle equal to the chosen angle of attack is laid off from each of these lines, and the new lines obtained represent the chords of the various sections.

It is to be observed that if the angle of attack of all elements of the blade is the same for a certain value of $V$ and $n$, it need not be the same for all elements for other values of $V$ and $n$.

Let the propeller be designed for a velocity $V_0$, and number of rotations $n_0$, and choose $\phi_0$ as the angle of attack of all elements; then the setting of an element is given by

$$\theta = \phi_0 + \tan^{-1} \frac{V_0}{2\pi n_0 r},$$

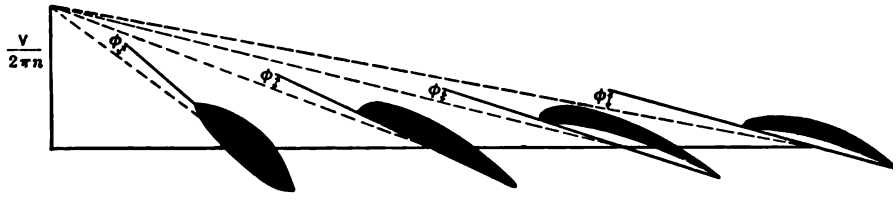


FIG. 34.

as a function of $r$. Now suppose this propeller advances at rate $V$ and makes $n$ turns per minute. The angle $\beta$ is now

$$\beta = \tan^{-1} \frac{V}{2\pi n r},$$

and the angle of attack is

$$\phi = \theta - \beta = \phi_0 + \tan^{-1} \frac{V_0}{2\pi n_0 r} - \tan^{-1} \frac{V}{2\pi n r}$$

$$= \phi_0 + \tan^{-1} \frac{nV_0 - n_0 V}{(2\pi r)^2 n n_0 + V V_0}.$$

It is seen that the necessary and sufficient condition that this does not depend upon $r$ is

$$nV_0 - n_0 V = 0,$$

that is,

$$\frac{V}{n} = \frac{V_0}{n_0}.$$

The angle of attack will thus again be the same for all elements of the blade, only if the advance per revolution is the same as it was before. This quantity is of fundamental importance in considerations regarding a propeller.

**66. Thrust and Power.**—In order to calculate the thrust that a propeller will produce, the power that is necessary to rotate it, and the efficiency with which it is working, experiments are made on model propellers, over a wide range of forward velocities, and speeds of rotation. From the data thus obtained the action of a full size, geometrically similar, propeller can be deduced. We proceed to a discussion of the conditions under which we can compare the action of similar propellers.

Consider geometrically similar propellers of diameters $D$ and $D_1$, respectively. Let them be advancing at velocities $V$ and $V_1$, and rotating $n$ and $n_1$ times a second, respectively. In case the angles of attack of corresponding sections of the two propellers are the same, we can compare the action of the two; and it is quite apparent that we cannot expect to find any simple relation between the action of the two propellers unless this is the case. The condition for the equality of the angles of attack of corresponding sections, always under the hypothesis that the propellers act on air at rest, is easily obtained. The settings of the corresponding sections are equal, since the propellers are geometrically similar. It is then necessary that the angle $\beta$ be equal in the two sections. The condition for this is obviously

$$\frac{V}{nr}=\frac{V_1}{n_1 r_1},$$

where $r$ and $r_1$ are the radii of the sections. This reduces to

$$\frac{V}{nD}=\frac{V_1}{n_1 D_1},$$

since $r/r_1=D/D_1$ for similar sections.

It is assumed that the propellers are acting under conditions that satisfy this relation.

In the two propellers take sections near those already chosen, the new sections also being at distances proportional

to the diameters. Consider the portions of the two blades thus obtained as small aërofoils. As they are engaging the air at the same angle of attack the air reactions will be assumed to make the same angles with their chords, and thus with the axes of the propellers. Let the air reactions be $dF$ and $dF_1$.

Then

$$\frac{dF}{dF_1} = \frac{ds \cdot V'^2}{ds_1 \cdot V_1'^2},$$

where $ds$ and $ds_1$ are the areas of the two sections of blade, and $V'$ and $V_1'$ their total velocities (resultants of $V$ and $2\pi nr$, and $V_1$ and $2\pi r_1 n_1$, respectively).

Now

$$\frac{ds}{ds_1} = \frac{D^2}{D_1^2},$$

in virtue of the similarity of the propellers, and

$$\frac{V'}{V_1'} = \frac{2\pi nr \sec \beta}{2\pi n_1 r_1 \sec \beta} = \frac{nr}{n_1 r_1} = \frac{nD}{n_1 D_1}.$$

Hence

$$\frac{dF}{dF_1} = \frac{n^2 D^4}{n_1^2 D_1^4}.$$

Let $dT$ and $dT_1$ be the elements of thrust furnished by the two sections. Since the reactions make equal angles with the axes, we have

$$\frac{dT}{dT_1} = \frac{n^2 D^4}{n_1^2 D_1^4}.$$

By dividing the two propellers up into corresponding small sections and adding the thrusts of the sections, we have

$$\frac{T}{T_1} = \frac{n^2 D^4}{n_1^2 D_1^4},$$

for the relation between the total thrusts.

This relation leads us to write

$$T = \alpha n^2 D^4,$$

where $\alpha$ is some function of $V/nD$, applicable to geometrically similar propellers. Such a relation must be confirmed by experiment. This question will be considered presently.

Let $dR$ and $dR_1$ be the resistance to rotation of the two sections. Then

$$\frac{dR}{dR_1}=\frac{n^2D^4}{n_1^2D_1^4}.$$

Let $dP$ and $dP_1$ be the work done on the two sections in a second; then

$$dP=2\pi nr\,dR, \qquad dP_1=2\pi n_1r_1dR_1.$$

Hence,

$$\frac{dP}{dP_1}=\frac{n^3rD^4}{n_1^3r_1D_1^4}=\frac{n^3D^5}{n_1^3D_1^5}.$$

We are thus led to put for the power necessary to turn the propeller

$$P=\beta n^3D^5,$$

where $\beta$ is a function of $V/nD$ applicable to geometrically similar propellers.

**67.** The results obtained can be put in another form. We have

$$T=\alpha n^2D^4=\alpha V^2\left(\frac{n^2D^2}{V^2}\right)D^2.$$

Since $\alpha$ is a function of $V/nD$, the quantity $\alpha(n^2D^2/V^2)$ is also a function of $V/nD$. Hence we are led to write,

$$T=\mathfrak{T}V^2D^2 \quad \text{(pounds)},$$

where $\mathfrak{T}$ is a function of $V/nD$.

In the same way we obtain

$$P=\mathfrak{P}V^3D^2 \quad \text{(foot-pound-seconds)},$$

where $\mathfrak{P}$ is a function of $V/nD$.

**68.** To investigate the accuracy of the results that have been obtained it is necessary to test several propellers that are geometrically similar. Consider one of them. The thrust $T$ is measured for a large range of values of $V$ and $n$. The quantity $T/V^2D^2$ is calculated and is plotted against the argument $V/nD$. In this way a curve is obtained that represents the value of $\mathfrak{T}$ for the first propeller. The other propellers are then tested, and it is seen how nearly identical the curves they give are with the first.

In the same manner the formula obtained for the power is subjected to verification.*

**69. Efficiency.**—The efficiency of a propeller is the ratio between the useful power it yields and the power that it absorbs from the motor. Let it be represented by $E$. Then,

$$E=\frac{T\cdot V}{550\cdot P},$$

the thrust $T$ being measured in pounds, and the velocity $V$ in ft./sec. and $P$ in horse-power. When the expression obtained for $T$ and $P$ in terms of $V$ and $D$ are used this becomes

$$E=E\left(\frac{V}{nD}\right),$$

$E$ being, as indicated, a function of $V/nD$, applicable to similar propellers.

**70. Effect of Altitude.**—The thrust that a propeller will produce and the power necessary to turn it vary directly as the density of the air, and will thus depend upon the altitude. It is apparent, however, that the efficiency is independent of the altitude.

The diagrams that will be given for a propeller's action are for the surface of the earth. The thrust and power for a given altitude will then be found by multiplying the thrust and power at the surface of the earth for the same velocity of advance and number of rotations per minute by the ratio of the air density at the given altitude to the density at the surface of the earth.

**71.** Graphs of the quantities $\mathfrak{T}$, $\mathcal{P}$, $E$ for a model propeller†

* Results of experiments of this sort on four geometrically similar propellers of diameters 30, 36, 42, 48 inches, respectively, are given in Report No. 14, part I, Figs. 11, 12, of the Third Annual Report of the National Advisory Committee for Aeronautics, 1917.

† This is propeller No. 1, in the Report of the Advisory Committee on Aeronautics above referred to. The curves are not given in the form in which they occur in the report. The density of the air at the surface of the earth is introduced. The constant 100 that occurs in the report is omitted from the thrust curve. The power curve is obtained from that for torque.

are given in Fig. 35. The quantities $\mathfrak{T}$ and $\mathcal{P}$ are called the thrust and power coefficients, respectively.

As an example of the use of the graphs, consider a full-scale similar propeller with a diameter of 8 feet. What will be its thrust and the horse-power required to turn it at an altitude of 2000 feet, if it is advancing at a velocity of 72 miles an hour, and making 1200 turns a minute?

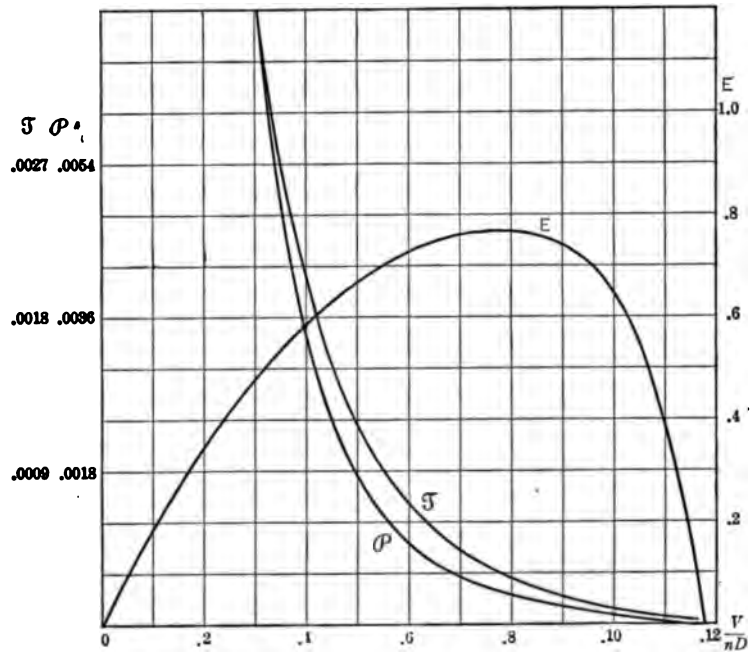


FIG. 35.

We have $V = 105$ ft./sec., $n = 20$. Hence $\frac{V}{nD} = .66$. From the diagram we find $\mathfrak{T} = .00054$, $\mathcal{P} = .0007$. The ratio of the density of the air at the given altitude to the density at the surface of the earth is .9. We thus have

$$T = .9 \times .00043 \times 105^2 \times 8^2 = 350 \text{ pounds},$$

$$P = \frac{.9 \times .0007 \times 105^3 \times 8^2}{550} = 85 \text{ horse-power}.$$

The efficiency can be immediately obtained from the graph, and is $E = .75$. It can also be computed directly from the values for $T$ and $P$.

**72.** A problem of fundamental importance is that of the adaptation of a propeller to a given airplane and motor. This question can be well studied by constructing a series of performance curves for the propeller.

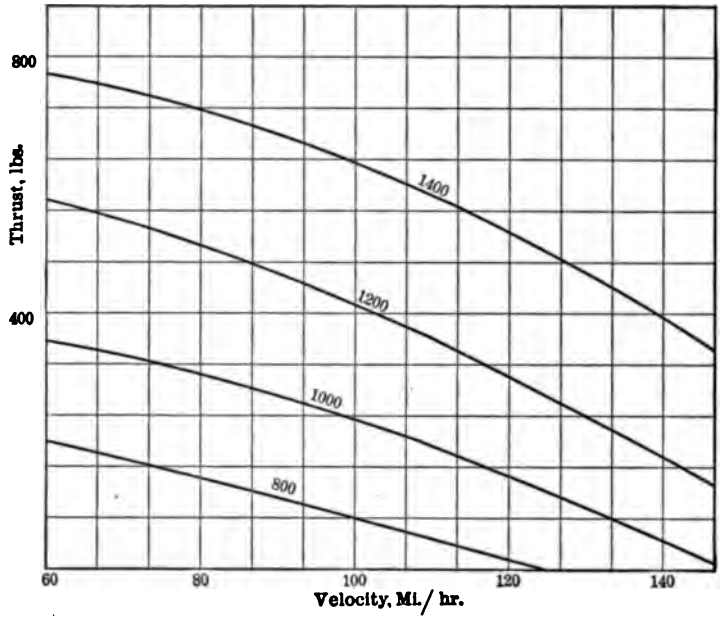


FIG. 36.

Consider the propeller of 8 feet diameter used in the last section, and let it be acting at the surface of the earth. By giving to $n$ successively the values 800, 1000, 1200, 1400 revolutions per minute, the thrust and power can be plotted against forward velocity.

The results for the thrust are shown in Fig. 36. It is seen that:

1. *For a given number of revolutions per minute the thrust decreases with increasing velocity of translation.*

2. *For a given velocity of translation the thrust increases with an increasing number of revolutions per second.*

These results are to be expected. For a reference to Fig. 32 shows that for $n$ fixed, increasing $V$ diminishes the angle of attack of the various elements of the blade, and hence decreases the resultant air reaction, and consequently the thrust, since the total reaction is approximately normal to the chord. On the other hand, for a fixed $V$, increasing the value of $n$ will have the opposite effect.

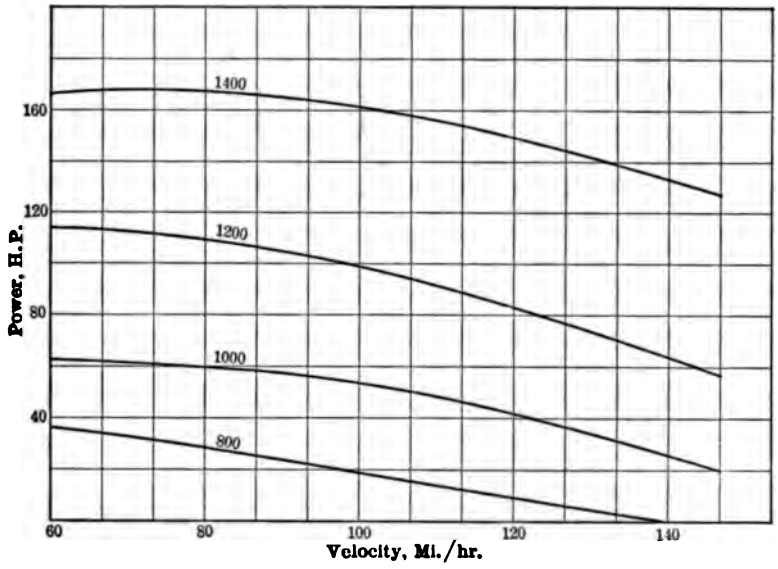


FIG. 37.

The curves representing the power are shown in Fig. 37. It is evident that:

1. *For a given velocity of rotation the power absorbed by the propeller decreases with increasing forward velocity.*

2. *For a given velocity of advance the power necessary increases with increasing rotational velocity.*

The efficiency can be found from Fig. 35. For convenience of reference the following table of values for $V/nD$ for the values of $V$ and $n$ used is given:

| V / n | 800 | 1000 | 1200 | 1400 |
|---|---|---|---|---|
| 60 | .56 | .45 | .37 | .32 |
| 70 | .66 | .52 | .44 | .37 |
| 80 | .75 | .60 | .50 | .43 |
| 90 | .84 | .67 | .56 | .48 |
| 100 | .94 | .75 | .62 | .53 |
| 110 | 1.03 | .82 | .69 | .59 |
| 120 | 1.13 | .90 | .75 | .64 |

**73. Motor Diagram.**—In order completely to solve the problem of adaptation of the propeller to the machine, it is necessary to know the manner in which the power of the motor varies with the number of revolutions. Up to a certain limit

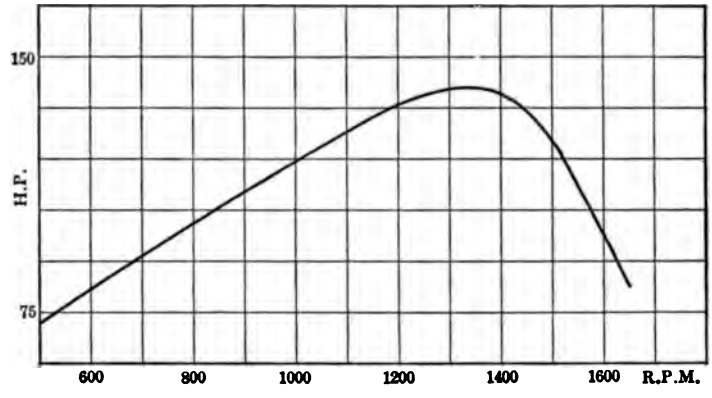


FIG. 38.

the power of the motor is approximately proportional to the number of explosions per minute, that is, to the number of revolutions. After that, due to choking, the power decreases rapidly with the number of revolutions. A diagram giving a motor's performance is as shown in Fig. 38. This diagram is constructed for full admission. When the motor is throttled a similar curve is obtained, lying below that given.

**74.** Consider now the conditions under which a given propelling plant will propel a given airplane. Suppose we wish the machine to be flown horizontally near the earth at a velocity of 80 miles an hour. In § 22 we described the method of

plotting the traction necessary against forward velocity. Suppose the traction is 400 pounds. From the thrust diagram for the propeller, we find the number of revolutions per minute that the propeller must be making in order that it shall produce a thrust of 400 pounds when advancing at 80 miles an hour. Suppose the value of $n$ thus found is 1100. We next turn to the power diagram and find what power must be furnished to the propeller. Suppose it is 85 horse-power. We finally consult the motor diagram and see what power it will furnish with full admission if running at 1100 revolutions per minute. If this is in excess of 85 we can, by throttling the engine properly, supply to the propeller exactly the power which will make it turn with the proper number of revolutions that will give it the forward velocity and traction that are necessary to sustain the machine.

**75.** While the curves which have already been given allow us to answer questions relative to the adaptability of a propeller to a given motor and airplane, it is possible to construct a diagram which will give a more comprehensive view of the problem.

On the power curves we construct curves of equal efficiency. For example, let us construct the curve representing an efficiency of 60 per cent. For $n=800$, say, we find from Fig. 35 the values of $V$. On the power curve for 800 rev./min. we locate the points corresponding to the values of $V$. We do this for $n=1000$, 1200, 1400, etc., and through the points found draw curves. This gives us the 60 per cent efficiency curves. Similarly, we construct the curves for $E=65$ per cent, 70 per cent, 75 per cent. Assuming that we do not desire any regime of operation where the efficiency falls below 60 per cent, we have a diagram which gives clearly the combinations of $V$ and $n$ which will give us an efficiency of the desired amount.

**76.** On this diagram we next construct a curve which represents the functioning of the motor for full admission. To do this, consider the performance diagram for the motor. For a given value of $n$ read the power the motor will furnish, and locate in Fig. 39 on the curve for the same value of $n$ the point

where the propeller absorbs that same power. This gives us a curve which intersects the two limiting efficiency curves already drawn. Under normal conditions motor and propeller

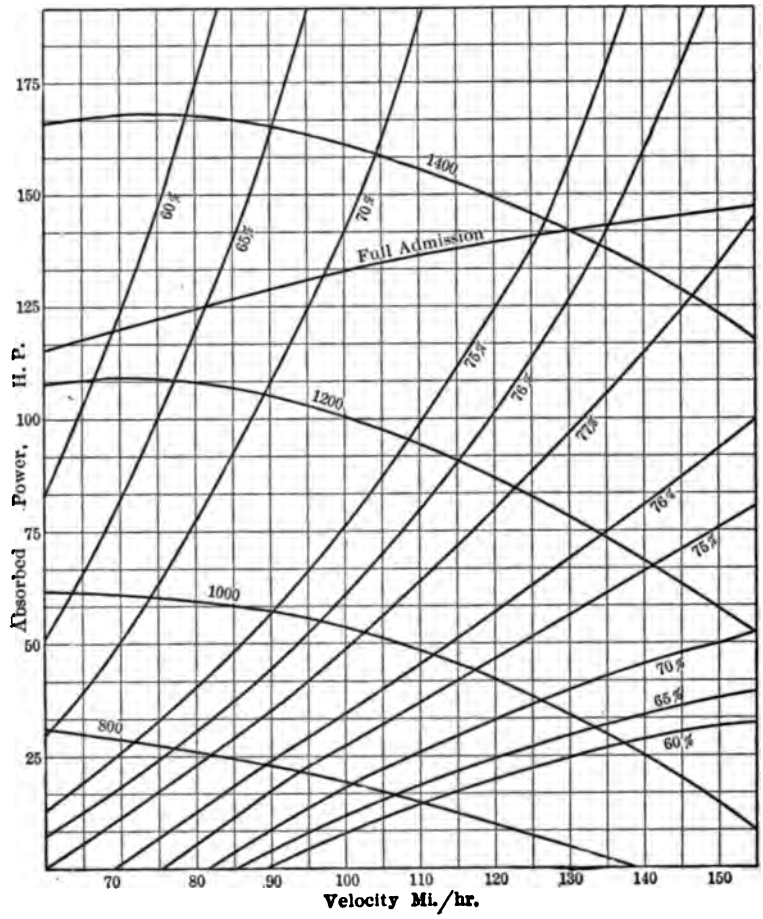


FIG. 39.

must be made to operate for values of $V$ and $n$ that lie within the triangular area which is determined in this way.

**77.** From the diagram last made it is easy to construct a diagram that gives the maximum useful power available from the motor-propeller group. Multiply the value of the power

along the full admission curve by the value of the efficiency, and plot the product against the value of $V$. This gives the curve $P_M$ shown in Fig. 40. For a given value of $V$ this curve

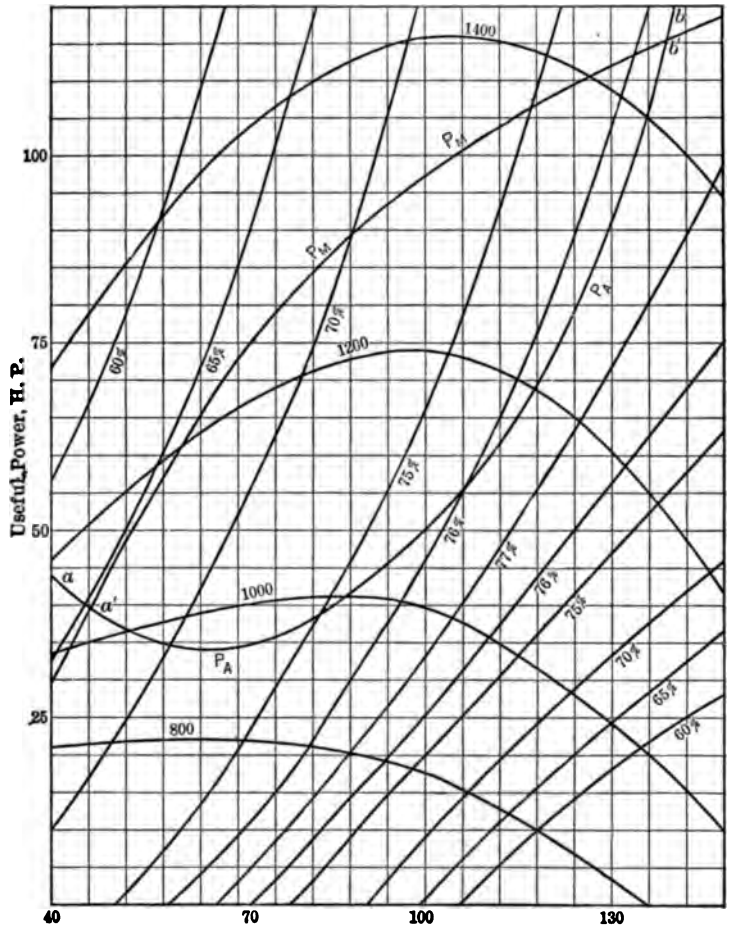


FIG. 40.

gives us the greatest useful power we can obtain from the propeller attached to the given motor. The efficiency curves are carried over from the last diagram in an obvious way. Also curves representing the number of revolutions per minute are easily constructed.

**78.** To answer the question of the adaptability of the motor and propeller to the airplane, we plot on the last diagram the useful power necessary to propel the airplane, as a function of $V$ (see § 24). This gives us the curve $P_A$. We plot only that amount of it within which operation would be safe. Call the extremities of the curve $a$ and $b$. The curve will, in general, intersect $P_M$ in two points $a'$ and $b'$. Thus the motor-propeller with full admission will propel the machine in horizontal flight at two different velocities. The degree of adaptability of the motor and propeller to the airplane depends upon the relative positions of $a$ and $a'$, $b$ and $b'$. If $b$ is beyond $b'$, as shown, the motor and propeller are unable to get the speed out of the machine that its construction would safely allow. If $b'$ were beyond $b$, the motor and propeller could develop a velocity that would be dangerous.

The greatest velocity of ascent will be attained when the excess power available is a maximum. This corresponds to a velocity of about 90 miles an hour in the case represented by the diagram given.

# CHAPTER VI

## PERFORMANCE

### 1. Ceiling

**79.** In § 48 we have given a discussion of the height to which a machine can fly. That height, which we called the ceiling, depends upon the angle of attack. The maximum of these heights for all angles of attack we called the true ceiling of the machine. A reference to § 44 would lead us to expect that the true ceiling would be reached when the angle of attack is the angle of minimum power, for then it would seem that there would be the greatest surplus power available for climbing. This conclusion depends, however, upon the supposition of constant available power, a condition, which as indicated in § 44, does not exist. As a matter of fact, the true ceiling is attained for an angle of attack more nearly equal to the optimum angle, and sometimes even smaller.* In order to determine the height of the ceiling by the method given in § 48, it is necessary to make an ascent of some height. We shall now consider the question of determining the height of the ceiling directly from the known motive power of the machine, its fineness, etc.

**80.** If the ceiling is to be as high as possible for a given machine and motor, it is necessary that the propeller be well adapted to the two. For suppose that the machine has risen to the greatest possible height. The motor will be running with a certain definite number of revolutions per minute. If this number is not that which gives the greatest possible power from the motor, and the value of $V/nD$ is not that which gives

* The Sorbonne lectures of Professor Marchis, Spring semester, 1919.

the greatest efficiency to the propeller, it is evident that there is a lack of proper adaptation in the various elements of the machine.

In the calculations that we shall make of the theoretical ceiling in terms of the motor power, propeller efficiency, and fineness of the machine, we shall assume that there is a complete adaptation. We shall therefore obtain a quantity which will in practice exceed the performance of the machine.

Let $P_0$ be the power of the motor at the ground, in horsepower, and $\mu_z$ the ratio of the height of the barometer at altitude $z$ to its height at the surface of the earth.* Then the motor power at altitude $z$, in foot-pound-seconds, is

$$P_z = 550 \cdot P_0 \cdot \mu_z.$$

Let $E$ be the propeller efficiency, which, of course, varies as the machine rises. But in accordance with what was stated above we shall assume that it has attained its maximum value at the time the machine ceases to rise. The useful power available at the ceiling is therefore

$$P_a = 550 P_0 \cdot \mu_z \cdot E.$$

The traction necessary for horizontal flight is $BW$, where $B$ is the fineness, and $W$ the weight. Therefore the power necessary is

$$P_m = B \cdot W \cdot V_h,$$

where $V_h$ is the horizontal velocity, in feet per second, at the ceiling.

Consequently, at the ceiling we have

$$550 \cdot P_0 \cdot \mu_z \cdot E - B \cdot W \cdot V_h = 0.$$

The fineness $B$ is independent of the altitude, but

$$V_h = \frac{V_0}{\rho_z^{1/4}},$$

* The remainder of this section is taken directly from the lectures of Professor Marchis.

where $V_0$ is the horizontal velocity at the surface of the earth, and $\rho_z$ the ratio of the density of the air at altitude $z$ to the density at the ground.

Inserting the value of $V_h$ and dividing by $P_0 \cdot E$, we have

$$\mu_z - \frac{B \cdot W \cdot V_0}{550 P_0 \cdot E} \cdot \frac{1}{\rho_z^{\frac{1}{2}}} = 0,$$

as the relation which must be satisfied at the ceiling.

If we put

$$M = \frac{B \cdot W \cdot V_0}{550 \cdot P_0 \cdot E},$$

we have a number characteristic of the complete machine. Into it enters the power of the motor, the efficiency of the propeller, the total weight of the machine, the fineness, as well as the horizontal velocity at the ground. It would again seem from this expression that the angle of attack to reach the ceiling should be that which renders $BV_0$ a minimum, that is, the economical angle. But on account of the considerations adduced in § 79, the fineness and velocity for the optimum angle are used.

If we give to $M$ a succession of values, such as .8, .6, .4, .3, .2, which will cover those that occur with customary machines, and for each such value of $M$, compute the value of the quantity

$$\mu_z - \frac{M}{\rho_z^{\frac{1}{2}}},$$

for a succession of values of $z$, differing, say, by 1000 feet, until we reach a value of $z$ for which the quantity in question is zero, we shall have the values of the ceiling that correspond to the various values of $M$. In this way we have a table that will give the theoretical ceiling for a machine, as soon as we know its characteristic number.

If, for example, we take $M = .4$ we find for $z = 17{,}000$ feet,

$$\mu_z - \frac{M}{\rho_z^{\frac{1}{2}}} = .526 - .4 \times 1.308 = .003.$$

Therefore we can take 17,000 feet to be the approximate theoretical ceiling.

Another remark may be added. The weight $W$ includes the weight of fuel. This continually decreases, so that $M$ decreases. The ceiling thus continually increases as the time increases, until such part of the fuel remains as the pilot desires to have at his disposition in the descent. If we include in $W$ the weight of the normal fuel carried, the calculation we have made gives the approximate height to which the machine rises before its course becomes sensibly horizontal.

**81. Supercharge.***—If it is desired to make a long flight, a quantity of fuel in excess of the normal is added. This increases $W$ and thus $M$, and therefore lowers the ceiling. A similar condition exists if any extra load is carried, though in the first case the surplus load continually disappears, while in the second instance that is not the case, unless perhaps the machine is such a machine as a bomber.

We shall investigate the change in the height of the ceiling produced by a supercharge. At the ceiling we have, as the equation of power,

$$B \cdot W \cdot V_h = 550 \cdot E \cdot P_0 \cdot \mu_h.$$

The equation of sustentation gives us

$$W = \rho_h K_y A V_h^2.$$

Let us assume that $\rho_h = \mu_h$ as an approximation.† Dividing the two equations we find

$$V_h^3 = \frac{550 \cdot E \cdot P_0}{B \cdot K_y A}.$$

If we assume that $E$ is constant, we see that $V_h$ is constant. Therefore the velocity at the ceiling is independent of any supercharge in weight that the machine may be carrying. A reference to the equation of power then gives

$$\frac{W}{\mu_h} = \text{constant}.$$

* Devillers, loc. cit., pp. 173-177.

† For the accuracy of this approximation see the Appendix, § 4.

Suppose now that $W$ represents the normal weight of the machine and $h$ its normal ceiling. Let the machine be supercharged to a total weight $W'$, and let $h'$ be the ceiling for this weight. We have

$$\frac{W'}{\mu_{h'}} = \frac{W}{\mu_h}.$$

Therefore

$$\mu_{h'} = \mu_h \frac{W'}{W}.$$

As all the quantities on the right are known, this equation will determine $\mu_{h'}$ and consequently $h'$.

For example, suppose a machine with a normal weight of 1500 pounds has a ceiling of 20,000 feet. Let the machine be supercharged to weigh 2400 pounds. What will be the new ceiling? We have $\mu_h = .468$. Hence,

$$\mu_{h'} = .468 \frac{2400}{1500} = .748.$$

The new ceiling is then approximately at 8000 feet.

### 2. Radius of Action *

**82.** Another problem of interest is that of calculating the distance to which a machine can fly. In addition to the other characteristics that must be taken into account we must now consider the total amount of fuel carried and its rate of consumption. Assumptions must again be made that will render the results only approximately correct, but nevertheless they will be of value as a basis of estimates as to performance.

Let $P_t$ = power of the motor at time $t$ ($t$ = 0 at the start);
$Q$ = total weight of fuel, gasoline and oil, at time of departure;
$Q_t$ = weight of fuel at time $t$;
$m$ = weight of fuel consumed per horse-power per hour;
$W$ = weight of machine at time of departure;
$B$ = fineness of the machine;
$W_t = W - Q_t$ = weight of machine at time $t$.

* Devillers, loc. cit., Chapter XII.

Assuming that the rate of consumption of fuel per horsepower is independent of the altitude, we have

$$dQ_t = \frac{mP_t}{3600} dt,$$

as the consumption in time $dt$, the second being taken as the unit.

The useful power is given at any instant by $W_t \cdot B \cdot V$. In order to obtain the power in terms of the performance of the motor we must assume a constant efficiency for the propeller. We shall take it to be .75.

The useful power developed by the motor in foot-pound-seconds is therefore $.75 \times 550 \times P_t$. We consequently have

$$.75 \times 550 \times P_t = W_t \times B \times V.$$

Hence,

$$P_t = \frac{W_t \cdot B \cdot V}{.75 \times 550},$$

and therefore,

$$dQ = \frac{mW_tBV}{3600 \times .75 \times 550} dt.$$

$$= \frac{mW_tBV}{1{,}365{,}500} dt.$$

Let $L$ represent the horizontal distance traversed. The machine in reality will be continually rising, due to decrease in weight from fuel consumption. However, except at the start, it will be practically horizontal. We take therefore $dL = Vdt$. The relation between the element of path and fuel consumption is therefore

$$dL = \frac{1{,}365{,}500}{mB} \cdot \frac{dQ_t}{W_t} = \frac{1{,}365{,}500}{mB} \cdot \frac{dQ_t}{W - Q_t}.$$

Integrating, and noting that at the start $Q_t = 0$, and at the instant of arrival $Q_t = Q$, we have

$$L = \frac{1{,}365{,}500}{mB} \log_e \frac{W}{W - Q}.$$

It is apparent from this that the greatest distance can be covered with a given amount of fuel if flight takes place at

the angle that gives the smallest value to $B$. Therefore the optimum angle should be used. During the flight the machine consequently flies at its ceiling, § 79.

We shall assume $m = 5.5$ and $B = .12$. When we change to common logarithms and to miles we find:

$$L = 9000 \log_{10} \frac{1}{1 - \frac{Q}{W}}, \text{ approximately.}$$

From this it is apparent that the distance that a machine can fly, under the suppositions we have made as to efficiency of propeller, fineness, and consumption of fuel, depends only on the ratio of the weight of fuel with which the flight is started, to the total weight at the time of departure.

In order to cover a great distance with a machine without replenishment of fuel it is, of course, necessary to greatly supercharge the machine at the start. This lowers the ceiling, as was stated in the last section, and it is necessary to know that the ceiling has not been lowered to such a point that flight would be dangerous.

Let us assume that a flight of 3000 miles is desired. Hence,

$$\log_{10} \frac{1}{1 - \frac{Q}{W}} = .3333,$$

from which

$$\frac{1}{1 - \frac{Q}{W}} = 2.15,$$

and therefore

$$\frac{Q}{W} = .54.$$

The weight of fuel carried must then be approximately equal to the weight of the remainder of the machine.

Suppose the values of $m$ and $B$ for a machine are not those used to obtain the formula for $L$. We should then multiply the value of $L$ given by the formula by $\frac{5.5}{m} \times \frac{.12}{B}$.

**83.** Let us now consider a flight that consists in going a certain distance and returning to the starting point, the complete flight to consume the entire fuel supply. It is evident that more fuel will be consumed in the going part of the trip than in the return. We desire to know the amount of fuel that can be consumed in the first part of the journey, and leave assured the possibility of return.

Let $W_0$ be the weight of the machine without any fuel (dead weight), $Q$ the total weight of fuel at the start, $Q_g$ the part of the fuel that will be expended in going, and $Q_r$ the part expended in returning. The going part of the trip can be thought of as that of a machine with dead weight of $W+Q_r$ and a fuel load of $Q_g$. The returning trip can be regarded as that of a machine of dead weight $W_0$ and fuel load $Q_r$. As the distances traversed are the same we have from the formula of the last section,

$$\frac{Q_r}{W_0+Q_r}=\frac{Q_g}{W_0+Q_r+Q_g},$$

which can be written

$$\frac{Q-Q_g}{W-Q_g}=\frac{Q_g}{W}.$$

From this we find

$$Q_g^2-2WQ_g+WG=0.$$

Therefore,

$$Q_g=W-\sqrt{W^2-WQ}=W\left[1-\sqrt{1-\frac{Q}{W}}\right],$$

since the only admissible root is one less than $W$.

If we develop the expression for $Q_g$, we find we can write

$$\frac{Q_g}{Q}=\frac{1}{2}+\frac{1}{8}\cdot\frac{Q}{W}+\ldots,$$

which gives the excess over half the total fuel which can be consumed before the return trip is started.*

* The method of § 83 can be extended to determine the performance of a bombing plane, or a machine carrying freight. In such cases there is a definite and considerable decrease of weight at a certain point in the journey, for instance the place where the return trip starts. See Devillers, loc. cit., pp. 185, 186.

## CHAPTER VII

### STABILITY AND CONTROLLABILITY

**84.** We come now to the consideration of a question which is of the greatest importance in the actual practice of flying, and of the greatest interest as a problem in dynamics. This question is that of stability and controllability. We shall begin the discussion by setting forth some general ideas involved in the problem, so as to make clear exactly what it is that is desired.

In a broad way the term stability relates to those properties which an airplane must possess in order that it be "air-worthy," that is, that it will be able to fly without too great an element of insecurity and danger. When we recall the fundamental principle by which flight is possible, namely, an equilibrium between tractive force of a propeller, air resistance on the machine, and gravity, and remember that the air resistance is delicately dependent upon speed and aspect of the machine with reference to the wind, while gravity is a force entirely beyond our control, we see that the problem will be one into which a thorough inquiry must be made. Once the equilibrium between the forces is destroyed, will it be possible to re-establish it? The air is never absolutely calm; and at different altitudes and in different localities, eddies and gusts of different natures and varying magnitudes will be encountered, so that flight under the ideal conditions that we have thus far assumed will never exist. Therefore, if secure flight is possible, it will be accomplished by a more or less frequent recurrence of states of losing and regaining equilibrium.

**85.** As the question turns primarily on the air reaction, we can seek to maintain equilibrium between the forces in two

general ways. We can construct the machine in such a way that it will inherently possess properties that will make it stable, and we can equip it with movable control surfaces that enable the pilot to alter the air forces and thus assist in re-establishing the equilibrium. These same control surfaces will also be the means by which the pilot maintains command of the machine, directing it along the course that he desires to follow. Thus we have the two qualities, stability and controllability, both necessary conditions, related and dependent upon each other, and as we shall see, to a certain extent incompatible.

**86.** In order to be inherently stable a machine must automatically maintain the same attitude towards the relative wind, for this will tend to keep the air reaction unaltered. When gusts are met, the machine will thus tend to head into them, and if this tendency is very strong it may be difficult for the pilot to keep the machine flying in the course he desires. In a general way, the greater the inherent stability the more difficult it will be to make the machine respond to the controls. It will tend to combat changes in its course. In some instances a great sensitiveness to controls is a necessity. Thus in the case of a battle plane, where rapid maneuvering is essential, the pilot must be able to get a quick and pronounced effect by moving his controlling surfaces. The degree to which a machine possesses inherent stability accordingly depends upon such things as size and purpose. While it is a quality that all machines must possess to a certain extent, still a condition can exist that could be described as "too stable." The machine then could be controlled only at the expense of considerable fatigue to the pilot, and furthermore, it might be an uncomfortable vehicle, owing to rapid oscillations that would arise when it encountered gusts.

**87.** In order to deal with the problem of stability by mathematical analysis, greater precision must be given to our definition. In fact, we can distinguish between stability in the general sense that has been considered, and in the special restricted mathematical sense to which we proceed, and which

will be developed to some extent in the next chapter. There can easily be a difference of opinion as to the degree in which it is necessary, or even desirable, that a machine should possess mathematical stability. On the other hand, it is agreed that a machine which is mathematically unstable in certain respects would be a very unsafe and undesirable one.

**88.** The question of stability is one that can enter into most dynamical situations. It relates to the effect of a displacement from a position of equilibrium. Such a displacement gives rise to changes in the forces acting. If the effect is to restore the state of motion to that which existed prior to the disturbance, the equilibrium is said to be stable; if the system tends to depart farther and farther from the prior state, the equilibrium is said to be unstable; if the system is indifferent to the change, and maintains a state of motion near to that which would have existed if the disturbance had not taken place, the equilibrium is said to be neutral. As an example of stable and unstable equilibrium, consider the motion of a marble down a hollow inclined pipe. Let it be on the inside. It will follow the lowest element, and if, in its motion, it be slightly deflected, it will, after oscillating back and forth, regain its former state of motion: its equilibrium is stable. Suppose, on the other hand, that the marble is on the outside of the pipe, and rolling down the top, which we shall consider slightly flattened. A small disturbance will cause it to depart from its path and fall from the pipe: the equilibrium is unstable. As an example of neutral equilibrium, let the marble roll down an inclined plane. A small lateral disturbance will cause it to follow a course near that it would have followed had it not been deflected: its equilibrium is neutral.

**89.** The example of stability which has been given illustrates two distinct ideas that enter into stability. After the displacement has taken place, there must exist a force tending to restore the former condition. If such a force exist, there is said to be **static stability.** The effect of this force will in general be such as to make the system with which we are dealing return to the position of equilibrium, and then depart

from it in the opposite sense. In this way oscillations will arise. If these oscillations die out as the time increases, there is said to be **dynamic stability.**

We shall in this chapter give a development of the simpler aspects of the problem of stability of an airplane, mainly those connected with static stability. The consideration of dynamic stability is reserved for the next chapter. A general understanding of the means by which stability and controllability are obtained can be secured by a simple analysis. It is only by such a preliminary procedure that we can construct models and make the proper and elaborate experiments that will furnish the data for an investigation of dynamical stability.

**90.** In our first analysis we shall consider the effect of rotation upon an airplane. To distinguish different types of rotation we draw three axes in the machine meeting at the center of gravity, one perpendicular to the plane of symmetry, one in the plane of symmetry, parallel to the propeller axis, for instance, and the third perpendicular to these two. A rotation about the first axis is described as **pitching**, about the second as **rolling**, about the third as **yawing**. We shall consider the effect on stability of rotations of these three types. The general rotation that a machine can experience is a combination of all three. The more elaborate investigation in the next chapter covers that case. It is, however, by considering separately the three types of rotation that one is led to develop principles of design that make stability possible.

We shall first consider the effect of pitching, which gives rise in its simplest form to what is called the problem of longitudinal stability.

## Longitudinal Stability

**91.** We can say in an approximate way that longitudinal stability depends primarily on the manner in which the center of pressure moves as the angle of attack varies. Consider a plane surface. Suppose it were the sustaining member of a machine. In horizontal flight the resistance $R$, the traction $T$,

and the weight $W$ meet at a point, say, the center of gravity. Suppose some influence tended to rotate the machine in such a way as to increase the angle of attack. The resistance $R$ recedes from the attacking edge, remaining sensibly parallel. A moment then arises that tends to return the machine to its original position. It is therefore statically stable.

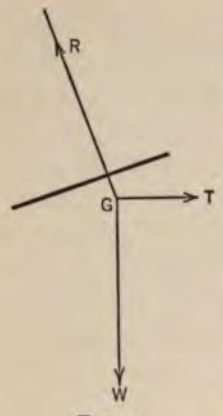

FIG. 41.

Suppose now that we had for sustaining member a single cambered surface. With increasing angle of attack the center of pressure tends, as a rule, to approach the edge of attack, and vice versa. Thus the air pressure tends to create a couple that will increase the displacement, and we would expect the machine to be statically unstable.

**92.** An analysis of this sort should be pursued further. Consider the surfaces of a biplane. For different angles of attack suppose that the line of action of the resultant air pressure were drawn in the plane of symmetry. This gives a one-parameter family of straight lines. They envelop a curve, called the **metacentric curve**. For the angles of attack that are used, this curve is concave to the front, as shown in the figure. The point of tangency of the metacentric curve and the line representing the air reaction is called the **metacenter**. As the angle of attack increases, the metacenter describes the metacentric curve in the sense $A$ to $B$.

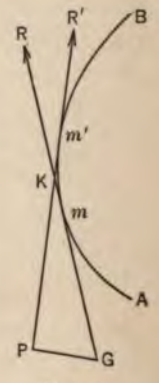

FIG. 42.

Suppose the machine is in horizontal flight. Let the propeller pass through the center of gravity $G$; then the machine being in equilibrium, the air reaction $R$ must also pass through $G$. Suppose the machine pitches so as to increase the angle of attack. The metacenter moves to $m'$, the line of action of

the air reaction to $R'$. A couple of moment $R' \times GP$ is created, where $P$ is the foot of the perpendicular from $G$ on $R'$, and $R'$ represents the magnitude of $R'$. This will be a restoring couple if $G$ is below $K$, the intersection of $R$ and $R'$; otherwise, it will tend to increase the displacement. Therefore there will be a condition of equilibrium if $G$ is below the metacentric curve. The metacentric curve, however, in general, lies too low for this condition to exist. Stability must therefore be obtained by some added feature. This feature is a **tail plane.**

**93. Tail Plane.**—Consider a plane surface to the rear of the sustaining surface. Let $ab$ be a section of it. Suppose that the air reaction on it is originally zero, that is, that it is in the bed of the wind. Let $G$ be the center of gravity. Suppose the machine is rotated about the center of gravity

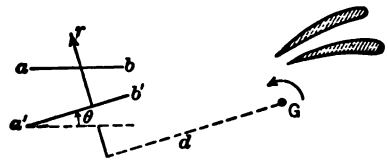


FIG. 43.

through an angle $\theta$; then the tail plane takes a position $a'b'$, inclined at an angle $\theta$ to the relative wind. The air reaction, $r$, on it will be practically normal to it, and can approximately be represented by $ks\theta V^2$, where $k$ is the constant given in § 3, and $s$ is the area of the tail plane. There thus arises a restoring moment of amount

$$ks\theta V^2 \cdot d,$$

where $d$ is the distance shown in the figure. The air reaction on the main sustaining surface likewise has a moment about $G$. The sum of the two must be a restoring moment, in order that the tail plane function in the desired manner. Similar considerations must hold for a motion that tends to decrease the angle of attack.

It is seen at once that a problem of great importance is the determination of the proper area for the tail plane, and its

distance from the sustaining surfaces. The angle at which it is set is also important. It is found that it must make a smaller angle with the relative wind than the main plane.* The tail plane does not act as an independent plane, but is greatly influenced by the wash of the main planes. Its actual behavior must be determined by experiment on different combinations. Investigations of this sort have been conducted by Eiffel, who has found that the wash from the front surface has the same effect as decreasing the angle of incidence of the tail plane. Thus, if the tail plane is apparently in the bed of the wind, it is in effect behaving as though it were at a negative incidence, and therefore has a downward pressure exerted upon it.

**94. The Elevator.**—For allowing control of the machine, and increasing longitudinal stability, it is fitted with an **elevator.** This is the plane attached to the rear of the tail plane, and

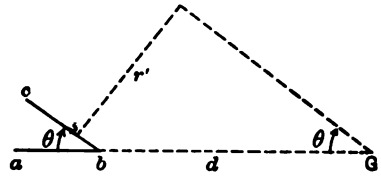


FIG. 44.

movable about a horizontal axis. By means of it the angle of attack of the machine is altered. Suppose the machine is in horizontal flight with the elevator in the neutral position *ab*. Let it then be turned through the angle $\theta$ into the position *bc*. There arises a force $r'$, which we can represent with sufficient accuracy by

$$r' = ks'V^2 \frac{\sin\theta}{.4+.6\sin\theta},$$

where $k$ is a constant and $s'$ the surface of the elevator. Suppose the center of gravity is at $G$. Let $Gb = d$; then assuming the

* Bothezat, "Étude de la Stabilité de l'aeroplane," p. 98.

force $r'$ as normal to the elevator, we have for the moment created:

$$M = \frac{k}{2} s' V^2 d \frac{\sin 2\theta}{.4 + .6 \sin \theta}.$$

The variation of this force with the value of $\theta$ is shown by a consideration of the function

$$\frac{\sin 2\theta}{.4 + .6 \sin \theta}.$$

This function reaches a maximum between 35° and 40°. It therefore follows that an elevator should not be given an inclination to exceed something like 30°.

The moment caused by turning the elevator into the position shown in the figure will have the effect of rotating the machine in such a direction as to increase the angle of attack. In the meantime the moment of the air reaction on the sustaining surfaces changes. The machine will rotate to a position such that the moment of all forces (propeller traction, air reaction on sustaining members, tail plane, elevator) passes through the center of gravity. If the motor power is at the same time properly altered, the machine will fly at a different angle of attack.

The means for changing the position of the elevator are secured by equipping it with a lever arm perpendicular to its surface, to the end of which a wire is attached, which in turn runs to the cock-pit. To turn the elevator may require considerable muscular effort on the part of the pilot. This effort can be lessened by properly balancing the elevator. The figure shows schematically the method of doing this. Let $aABb$ be the fixed tail plane, and $ABcde$ the elevator, pivoted about $AB$. It is

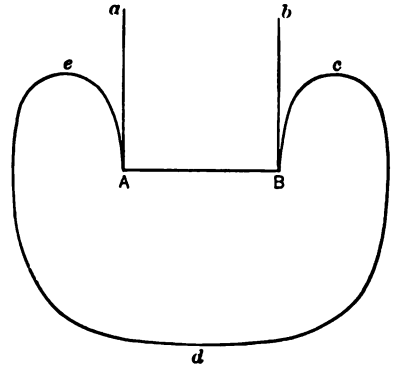

FIG. 45.

obvious that, by this disposition of the axis, the air reaction can be kept fairly near the axis so that the moment required to turn the elevator will be lessened. The resultant air reaction must not, however, in any position of the elevator pass through $AB$, for the pilot must always be able to feel a tautness in the controls.*

**95.** It is seen that we have considered only the question of static longitudinal stability, that is, the question of the existence of a couple tending to return the machine to its original position. Granting that such a couple exists we cannot ascertain the ultimate effect of the oscillations it will produce without knowing its magnitude as related to the angle through which the machine has been turned. We see also that while the rotation is going on, the instantaneous angle of attack of every element of the wing is changing, and the change of air reaction arising in this way is evidently dependent upon the velocity of rotation. It is possible to analyze with some approximations these various agencies that are at work, and obtain a differential equation from which we can draw conclusions as to the dynamic stability.† We shall not do this, however, for pitching is unavoidably connected with a change in the motion of the center of gravity, and we defer the whole question to the more accurate and complete discussion in the next chapter.

## Stability in Rolling

**96.** Suppose the machine possessed a single sustaining member whose leading and trailing edges were straight lines, that its body were that of a solid of revolution, that its tail plane were in the continuation of the axis of the body, that landing gear struts, wires, etc., were non-existent. Suppose that while in rectilinear flight with the axis of its body horizontal, it were made to roll about that axis. That half of the wing whose motion was downwards, would have its angle of attack increased; the half of the wing moving upwards, would have its angle of attack

* For a further discussion of this see Devillers, Chapter XIII.

† Devillers, Chapter XIII.

decreased. Consequently there would be a damping of the motion, dependent upon the velocity of rotation. Tail plane and rudder would also assist in this. When the motion had died out there would, however, be no restoring couple; that is, there is no static couple tending to restore the machine. It is obvious that during the motion, and in the displaced position, with the wings no longer horizontal, the vertical component of the lift no longer has its original value, and unless the proper variation of speed accompanied the process the altitude of the machine would change. But we are not here concerned with all the complications that arise. We merely are interested in seeing that there is no restoring couple.

**97.** While the discussion that has been given does not apply *in toto* to an actual machine, the general characterization can be carried over, and we see that we must provide a means of creating a restoring couple.

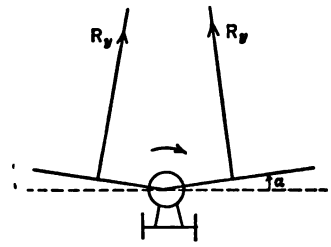

FIG. 46.

One way of causing stability in rolling is to set the wings at an angle, as shown in the figure. They are then said to possess a **dihedral**. The complete action of this is difficult to trace, but we can note one effect. For equilibrium we would have

$$2R_y \cos \alpha = W.$$

Now let the machine rotate through an angle $\theta$ so as to lower the left wing. The vertical component of the air reaction would be, assuming $R$ has not materially changed,

$$R_y \cos (\alpha - \theta) + R_y \cos (\alpha + \theta) = 2R_y \cos \alpha \cdot \cos \theta.$$

This is smaller than it was before. Such a rotation would then be accompanied by a downward motion of the machine. This would increase the lift on the left (lower) wing, more than on the right, and consequently a restoring moment would be created, that would make the machine roll back to its original position. The dihedral must not be too pronounced, or difficulty will arise from the lateral effect of a wind.

The fact that the dihedral produced stability through the consideration that rolling will produce a downward motion shows how impossible it is to separate completely different types of motion in the discussion of stability. They are all closely connected, and rotation about one line will cause rotations about other lines, though perhaps of a less pronounced nature.

**98.** Controllability in the lateral sense is furnished by means of the **ailerons.** Rectangular portions are cut from the corners on the trailing edge of the wings, and are then pivoted along their front edges. They are so fastened to the controls

FIG. 47.

that the raising of the ailerons on one side is accompanied by a lowering of those on the other. Their action is obvious. They merely increase the lift (and the drag) on one side and decrease it on the other. This gives the pilot the power of creating at will a moment that tends to roll the machine.

## LATERAL STABILITY

**99.** Lateral stability is secured by means of a **fin,** in the plane of symmetry, to the rear of the center of gravity. The general manner in which it functions is obvious. In case the machine is made to yaw, a moment is created tending to restore the machine. Of course, all parts of the machine have an effect in producing the restoring couple. Those parts well forward, such as the housing of the motor, tend to aggravate the yawing.

Controllability in direction is secured by means of a **rudder,** usually placed at the rear of the fin. In order to decrease the muscular exertion that the pilot must use to turn the rudder, it is generally balanced, as was explained in the case of the elevator. Here again it is necessary to be assured that

the force on the rudder will not pass through the axis, as in this case the pilot would not be sensible of any pressure on the controls. When the rudder is in the neutral position, it acts as a portion of the fin.

**100.** The complete analysis of the action of the rudder is difficult, for its turning introduces a sequence of phenomena hard to follow. The case is much more complicated than that of a ship, where the sustentation is in no way dependent upon speed.

Imagine that the rudder has been turned to the right. A force $f$ arises on the rudder, producing a moment tending to make the machine yaw to the right. To find the effect on the motion of the center of gravity we must apply this force at that point. We see then that the center of gravity moves slightly to the left at first. But the yawing produces a force $F$ on the opposite face of the fin. The effect of this transferred to the center of gravity will be to make the machine turn to the right. The total force tending to make the center of gravity describe a curve to the right is approximately $F-f$.

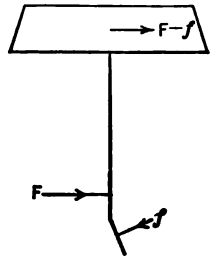

FIG. 48.

The total moment tending to produce rotation about a vertical axis is the resultant of the moments produced by the rudder and fin. It will be seen that a large fin well forward will make possible a considerable turning force by the use of a fairly small rudder.

The combined use of ailerons and rudder in turning has already been discussed in § 53.

**101.** It is not possible to separate a tendency to yaw from a tendency to roll, for the two will occur together. Thus, if the machine should start to yaw to the right, the left side of the wing will have a greater velocity than the right, and consequently a greater lift, so that the machine will start to roll.

The fin, while designed primarily for producing stability in direction, also contributes to stability in rolling, if properly placed. Suppose the machine had no dihedral, and that it

started to roll in the direction that causes the left wing to lower. The lift is no longer vertical, so the machine will start downwards. Suppose the machine had a vertical fin placed above the center of gravity. It will have been displaced from the vertical plane in the rolling, and when a tendency to settle begins, it is seen that a force arises on the fin that tends to make the machine roll back toward its original position. If the fin had been below the center of gravity, the effect on it would be to accentuate the roll. The efficiency of the fin in creating stability in rolling will also depend on its distance to the front or rear of the center of gravity.

In a similar way the dihedral of a machine adds to directional stability. For if the machine starts to yaw to the right, the effective angle of attack of the left wing is increased, that of the right wing decreased, and therefore a righting moment arises.

From the two properties that have been deduced, it appears that a dihedral in the wings is equivalent to a certain fin, and can for practical purposes be so regarded.

**102. Spiral Instability.**—A type of instability that is likely to occur, and is apt to prove very dangerous, is called spiral instability. It can be caused by large fin surfaces too far to the rear of the center of gravity. Suppose a machine with such a fin were banked so as to turn to the left. As the turn commences a pressure arises on the left side of the fin. This causes the machine to rotate so as to keep its axis along the path. The outside wing, moving faster than the inside, the lift is greater there, and this tends to increase the banking. Along with this goes a slipping towards the inside; for a shifting in the direction of the lift decreases the vertical force on the machine. This slipping causes a force on the fin, that tends to make the movement more pronounced. As a result of this sequence of phenomena the machine gets into a spin, or rapid nose dive, from which the pilot may be unable to extricate it.

## CHAPTER VIII

### STABILITY—(*Continued*)

**103.** A GENERAL discussion of the question of stability of an airplane can be made by the methods developed by Routh in his prize essay on the stability of a dynamical system.* The application of these methods to aviation was first given by Bryan.† His results are general, and the question of the stability of a machine for any slight disturbance from a position of equilibrium depends upon the nature of the roots of two biquadratic equations. The coefficients in these equations depend upon the construction of the machine. All elements that enter into the machine's construction, the wings, the stabilizing surfaces, the controlling surfaces, contribute to the value of the coefficients. In the application of the methods, difficulties arise in the determination of these coefficients. Bryan himself applies the method to machines of certain general characteristics, for which he can obtain with considerable certainty, values for the coefficients in terms of wing area, angle of attack, etc. Extensions and application of the method, by obtaining the values of the coefficients in the biquadratics through experiments on models in wind tunnels, has recently been made by other investigators.‡

**104. Moving Axes.**—Many problems in rigid dynamics are best treated by means of moving axes. These axes are fixed

* E. J. Routh, "Stability in Motion," London, Macmillan Co., 1877. See also his "Advanced Rigid Dynamics," Chap. VI.

† G. H. Bryan, "Stability in Aviation," Macmillan Co., 1911.

‡ Bairstow, "Technical Report of the (British) Advisory Committee for Aeronautics, for 1912–13," London, Darling & Son. In this report Bairstow gives simplifications in the equations of Bryan, and develops a method applicable to a machine without special hypothesis as to its construction.

in the body, move along with it, and assume continually changing positions and directions. They can afford us only indirectly a knowledge of how far the body has moved, but are peculiarly adapted to reveal the oscillations, and small changes in motion, that the body is experiencing at any instant. And in the question of stability it is exactly such points that are at issue. Use of moving axes was first made by Euler, and equations of motion for such axes were obtained by him. Bryan, however, found the particular system of coordinates used by Euler not well adapted to follow the motion of an airplane, and introduced slight changes in them.

**105.** Suppose that in a rigid body three axes be chosen, with their origin at the center of gravity. As the body moves, taking this system of axes with it, we fix our attention upon it at some particular instant. Its center of gravity has a vector velocity $V$, and the body a rotational velocity represented by another vector $R$. Resolve $V$ along the three directions in space that are *occupied at that instant* by the moving axes: let the components be $u$, $v$, $w$. Likewise resolve $R$: let its components be $p$, $q$, $r$. We thus obtain six functions of the time, and the equations of motion will be obtained by properly expressing these six functions and their derivatives with regard to the time in terms of the forces and moments acting on the body. Something as to the significance of the derivatives can be obtained by considering one of them, for instance $du/dt$. It measures the rate at which the component of a vector along a *varying* direction is changing. It is therefore *not* a component of the acceleration of the body, for a component of an acceleration is the rate at which the component of the velocity along some *fixed* direction is changing. Suppose that at instant $t$ the origin is at $O$, and let the position of the moving axis be $X$. Let $V$ be the vector velocity; then $u$ is the projection of $V$ on $X$. At a subsequent instant let the origin be $O'$ and the $x$ axis be denoted by $X'$, and the velocity by $V'$; then $u'$ is the projection of $V'$ on $X'$. Thus the increment of $u$ is $u'-u$. On the other hand, to find the acceleration of the body at the instant $t$ we must project $V'$ upon $X$, instead

of on the new position of that axis. The acceleration along the fixed direction in space, which at any instant coincides with the position of the moving axis, can, however, be expressed in terms of $du/dt$, $v$ and $w$. Similar remarks apply to the other quantities. We shall not derive these forms. They can be found treated by Routh.*

**106.** Choose for $x$-axis a line in the plane of symmetry, for instance, parallel to the propeller axis, directed backward;

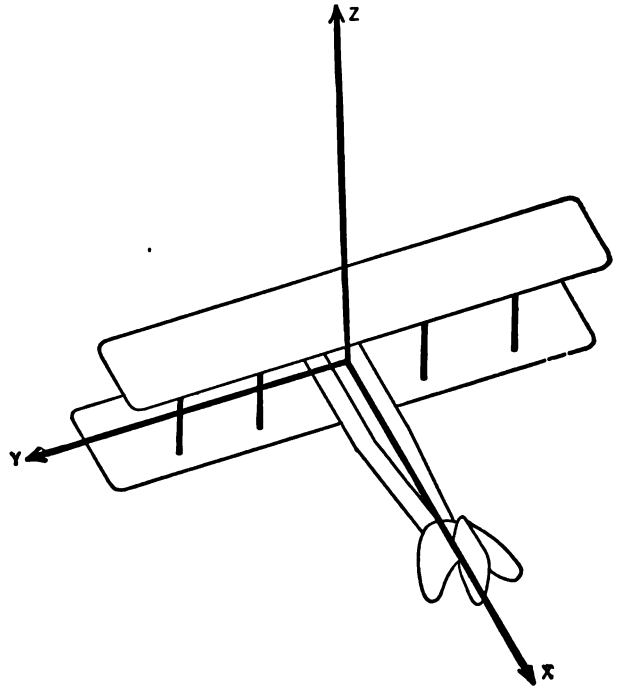


FIG. 49.

for $y$-axis, a line perpendicular to the plane of symmetry to the left as seen by the pilot; for $z$-axis, a perpendicular to these, in the plane of symmetry, directed upwards. The axes are further so chosen that the origin is the center of gravity of the machine.

* "Elementary Rigid Dynamics," Chapter V, "Advanced Rigid Dynamics," Chapter I.

With this choice of axes the accelerations of the center of gravity along fixed directions in space, that are the instantaneous positions of the moving axes are, respectively:

$$\frac{du}{dt}+wq-vr,$$

$$\frac{dv}{dt}+ur-wp,$$

$$\frac{dw}{dt}+vp-uq.*$$

We need also the rates of change of the angular momenta about the various axes. We denote these momenta by $h_1$, $h_2$, $h_3$; then

$$h_1=pA-qF-rE,$$
$$h_2=qB-rD-pF,$$
$$h_3=rC-pE-qD,$$

where $A$, $B$, $C$ are the moments of inertia, and $D$, $E$, $F$ the products of inertia. The machine being symmetric we have $D=F=0$. The rates of change of the angular momenta are then:

$$\frac{dh_1}{dt}-rh_2+qh_3,$$

$$\frac{dh_2}{dt}-ph_3+rh_1,$$

$$\frac{dh_3}{dt}-qh_1+ph_2.\dagger$$

**107.** It is necessary for us to have a means of comparing the orientation of the machine with some fixed orientation. This standard of reference we take as one in which the $xy$

* When comparison is made with Routh, "Advanced Rigid Dynamics," § 5, it will be found that the signs of the second and third terms are changed. This comes from the fact that Routh uses a right-hand system of axes while we have a left-hand system. In our system a rotation of an ordinary screw from the $x$-axis into the $y$-axis would advance the screw along the negative $z$-axis.

† Routh, "Advanced Rigid Dynamics," § 5.

plane is horizontal. From this position rotate the machine first about the $z$-axis through an angle $\psi$, then about the $y$-axis through an angle $\theta$, and finally about the $x$-axis through an angle $\phi$. A rotation about the $\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\}$ axis is positive when it turns the $\left.\begin{matrix} y \\ z \\ x \end{matrix}\right\}$ axis into the $\left.\begin{matrix} z \\ x \\ y \end{matrix}\right\}$ axis. The original positions of the axes are shown by the letters with subscripts o, and the posi-

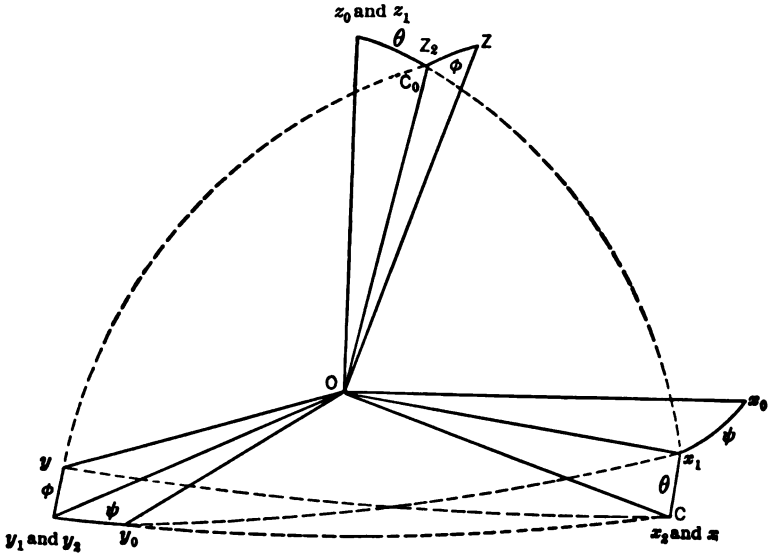


FIG. 50.

tion after the first and second rotations are shown respectively by subscripts 1, 2, and the final positions by the letters without subscripts. Further, any position of the coordinate axes can be obtained by a unique rotation of the sort described. Consider a position, represented by $x, y, z$. The angle between the planes $xz_0$ and $x_0z_0$ determines $\psi$. Let the plane $xz_0$ intersect the $x_0y_0$ plane in $Ox_1$; then $\theta$ is the angle between $Ox$ and $Ox_1$. Let $Oy_1$ be the intersection of the $x_0y_0$ and $yz$ planes; then $\phi$ is the angle between $Oy_1$ and $Oy$. To fix

the orientation of the machine at any instant we draw through the center of gravity axes parallel to those of reference, and determine $\psi$, $\theta$, $\phi$ as just described.

**108.** Suppose the orientation of the machine is continually changing; then $\psi$, $\theta$, $\phi$ are functions of the time. From their instantaneous values, and those of their derivatives with regard to the time, we can obtain the instantaneous values of $p$, $q$, $r$. We shall be in need of these relations, and will proceed to their determination.

About the origin in Fig. 50 draw a sphere of unit radius. Consider the point $C$. As the $x$, $y$, $z$ axes move, $C$ has a velocity $d\theta/dt$ along the arc $xz_0$, and a velocity $\cos\theta d\psi/dt$ normal to the plane $xOz_0$, that is, along $xy$. Now the angular velocity $r$ is the velocity with which $x$ approaches $y$, that is, the velocity along $xy$. But $xy$ makes an angle $\phi$ with $xy_1$. Resolving in the direction $xy$ the velocity $d\theta/dt$ along $z_0x$, and $\cos\theta d\psi/dt$ along $xy_1$, we have

$$r = -\sin\phi\frac{d\theta}{dt}+\cos\phi\cdot\cos\theta\cdot\frac{d\psi}{dt}.$$

Likewise $q$ is the velocity of $C$ along $zx$; hence

$$q=\cos\phi\cdot\frac{d\theta}{dt}+\sin\phi\cdot\cos\theta\cdot\frac{d\psi}{dt}.$$

Finally, $r$ is the velocity of $z$ away from $y$ along $yz$. But the velocity of $z$ relative to $C_0$ is $d\phi/dt$, and that of $C_0$ itself is $\sin\theta\cdot d\psi/dt$. Hence taking account of the positive directions for increasing $\psi$ and $\phi$,

$$p=\frac{d\phi}{dt}-\sin\theta\cdot\frac{d\psi}{dt}.*$$

When the angles $\psi$, $\theta$, $\phi$ are all zero, we have

$$p=\frac{d\phi}{dt}, \qquad q=\frac{d\theta}{dt}, \qquad r=\frac{d\psi}{dt},$$

* The values of $p$, $q$, $r$ are similar to the well-known Euler geometrical equations. See Routh, "Elementary Rigid Dynamics," § 256.

and when the angles are small, we can also use these values for $p$, $q$, $r$, with a sufficient degree of precision. The use of the exact values would introduce a very high degree of complexity into our work.

**109. Equations of Motion.**—The equations of motion are easily derived from the expressions that have been given in § 106. The motion of the center of gravity is determined by equating the expressions for the linear acceleration to the forces per unit mass acting along the respective axes. The rotation will be determined by equating the rates of change of angular momenta to the moments of the forces about the respective axes.

The forces acting come from three sources: gravity, propeller traction, and air reaction. The components along the axes of the force of gravity depend upon the orientation of the machine. Let $\psi$, $\theta$, $\phi$ be the values of the angles giving the orientation, as explained in § 107. Then the components of the weight per unit mass are:

$$g \cdot \sin\theta, \qquad -g \cdot \cos\theta \cdot \sin\phi, \qquad -g \cdot \cos\theta \cdot \cos\phi,$$

along the $x$, $y$, $z$ axes, respectively.

The moments due to the weight are zero.

Next, consider the traction of the propeller. Let it be parallel to the $x$-axis, of numerical value $H$ per unit mass, and applied at a distance $h$ above the $x$-axis. Its components are:

$$-H, \qquad 0, \qquad 0.$$

The moments of this force are:

$$0, \qquad -hH, \qquad 0,$$

about the $x$, $y$, $z$ axes, respectively.

There remains the air reaction. It depends upon the instantaneous velocity, translational and rotational; that is, upon $u$, $v$, $w$, $p$, $q$, $r$. No simple expression can be given for it. We denote the components of this force, per unit mass, at a given instant by $X$, $Y$, $Z$, and its moments about the respective axes by $L$, $M$, $N$.

The equations of motion now become

$$\frac{du}{dt}+wq-vr=g\cdot\sin\theta+X-H,$$

$$\frac{dv}{dt}+ur-wp=-g\cdot\cos\theta\cdot\sin\phi+Y,$$

$$\frac{dw}{dt}+vp-uq=-g\cdot\cos\theta\cdot\cos\phi+Z,$$

for the motion of the center of gravity, and (by making use of the angular momenta in terms of the angular velocities, remembering that the products of inertia $D$ and $F$ are zero),

$$A\frac{dp}{dt}-E\frac{dr}{dt}+(C-B)rq-Eqp=mL,$$

$$B\frac{dq}{dt}+(A-C)rp-(r^2-p^2)E=m(M-hH),$$

$$C\frac{dr}{dt}-E\frac{dp}{dt}+(B-A)pq+qrE=mN,$$

for the rotation about the axes.

**110.** We shall employ the equations of motion to determine the effect of a disturbance that a machine might undergo while it is in a state of steady motion. We limit ourselves to the simplest case, for even in this case the equations with which we must deal assume a sufficiently complicated form. The analysis of the subject which has been developed to date, will not enable us to treat a general displacement from a general position of equilibrium in which a machine might find itself. We can study some simple cases, those applying to normal states of flight. If we find in these instances a high degree of equilibrium, we will be led to infer that the machine can be regarded as air-worthy, and will furnish a means of transit in general of a sufficient degree of security.

Suppose the machine is in rectilinear horizontal flight. The $xz$ plane is vertical. Assume also that the $x$-axis is horizontal. (We have assumed the $x$-axis parallel to the pro-

peller axis. The new assumption is really not a restriction, as the equations that follow could easily be modified in such a way as to take care of the case where in the horizontal rectilinear flight the $x$-axis is inclined at a certain angle. There would, in general, be some angle of attack for the wings for which the $x$-axis is horizontal.) In this state of steady motion, $v, w, p, q, r$ are all zero, while $u$ has a constant value $U$ (which is negative on account of the direction in which the $x$-axis is chosen). The steady motion is represented by the equations:

$$\begin{aligned} 0 &= X_0 - H_0, \\ 0 &= Y_0, \\ 0 &= Z_0, \\ 0 &= mL_0, \\ 0 &= m(M_0 - hH_0), \\ 0 &= mN_0, \end{aligned}$$

where a letter with subscript zero denotes the value of that quantity for the state of steady motion.

Now imagine a disturbance to take place. Its exact nature we do not specify, nor do we know exactly what effect it will instantaneously have upon the machine. We shall merely say that it has altered all the velocities of the machine. Thus $v, w, p, q, r$ have values other than zero, while $u$ takes the value $U+u$. (The change in the significance of $u$ will cause us no confusion.) As is usual in the case of such discussions, we assume that $u, v, w, p, q, r$ are small enough that their squares and products can be neglected.

**111.** The orientation of the machine will be different. We take the orientation prior to the disturbance as the one of reference. After the disturbance the position of the machine will be represented by $\psi, \theta, \phi$, all of which we assume sufficiently small for us to write

$$p = \frac{d\phi}{dt}, \qquad q = \frac{d\theta}{dt}, \qquad r = \frac{d\psi}{dt}.$$

The disturbance will have changed all the components of the air reaction. Consider one of them, $X$. Putting in evidence the quantities upon which it depends we obtain

$$X(U+u, v, w, p, q, r) = X(U, 0, 0, 0, 0, 0)$$

$$+u\frac{\partial X}{\partial u}+v\frac{\partial X}{\partial v}+w\frac{\partial X}{\partial w}+p\frac{\partial X}{\partial p}+q\frac{\partial X}{\partial q}+r\frac{\partial X}{\partial r}.$$

That is,

$$X = X_0+uX_u+vX_v+wX_w+pX_p+qX_q+rX_r,$$

where $X_0$ is the value of $X$ prior to the disturbance, and $X_u$ $X_v, \ldots X_r$ are constants called "resistance derivatives," by Bryan. In all, we shall have thirty-six resistance derivatives, namely:

$$S_s, \qquad S = X, Y, Z, L, M, N,$$

$$s = u, v, w, p, q, r.$$

On account of the symmetry of the machine eighteen of the resistance derivatives vanish. Those that vanish are

$$S_s, \qquad S = W, Z, M,$$

$$s = v, p, r,$$

and

$$S_s, \qquad S = Y, L, N,$$

$$s = u, w, q.$$

To show that these eighteen resistance derivatives are zero, consider one of them, for instance, $X_v$. If it were not zero, a sideways velocity $v$, would cause an increase $vX_v$ in the force along the $z$-axis, and this would be positive or negative according as the displacement were towards the left or right, which could not be the case if the machine were symmetrical.

Consider also $Y_q$. If it were different from zero, a dip in the machine would produce a sideways force of a different direction from that produced by a tip.

Finally, let $H_0+\delta H$ be the new propeller traction.

The differential equations that represent the motion subsequent to the disturbance will be obtained by substituting the new values of the velocities in the six equations at the end of § 109, dropping products of small quantities, using the new values for the various forces and moments, and at the same time making use of the relations for steady motion.

We obtain thus:

$$\frac{du}{dt} = \theta g + \delta H + uX_u + wX_w + qX_q,$$

$$\frac{dv}{dt} + rU = -g\phi + vY_v + pY_p + rY_r,$$

$$\frac{dw}{dt} - qU = uZ_u + wZ_w + qZ_q,$$

$$A\frac{dp}{dt} - E\frac{dr}{dt} = m(vL_v + pL_p + rL_r),$$

$$B\frac{dq}{dt} = m(uM_u + wM_w + qM_q - h\cdot\delta H),$$

$$C\frac{dr}{dt} - E\frac{dp}{dt} = m(vN_v + pN_p + rN_r).$$

These six equations divide into two groups. The first, third, and fifth contain only $u$, $w$, $q$, and their derivatives; the second, fourth and sixth contain $v$, $p$, $r$ and their derivatives. The first group determines motion in the $xy$ place. Such motions are called **symmetric** or longitudinal oscillations. The second group determines oscillations that are called by Bryan the **asymmetric** oscillations.

**112.** As the equations contain also $\theta$ and $\phi$, we replace $p$ and $q$ by $d\phi/dt$ and $d\theta/dt$, respectively, and thus take $u$, $v$, $w$, $\phi$, $\theta$, $r$ as the quantities to be sought.

Finally, we take $\delta H = 0$; that is, we assume that the rotation of the propeller changes in such a way that the traction is unaltered by the disturbance.*

* For discussion of the case where this assumption is not made see Bryan, loc. cit., p. 28.

We make these substitutions, represent differentiation by the symbol $D$, and divide the equations into the two groups mentioned. For the longitudinal motion we have:

$$\begin{aligned}(D-X_u)u-X_w w-(X_q D+g)&=0,\\ -Z_u u+(D-Z_w)w-(Z_q+U)D\theta&=0,\\ -M_u u-M_w w+(k_B^2D^2-M_q D)\theta&=0,\end{aligned}$$

where we have put $k_B^2=B/m$; and for the asymmetric oscillations we have:

$$\begin{aligned}(D-Y_v)v-(Y_p D-g)\phi+(U-Y_r)r&=0,\\ -L_v v+(k_A^2D^2-L_p D)\phi-(k_E^2D+L_r)r&=0,\\ -N_v v-(k_E^2D^2+N_p D)\phi+(k_c^2D-N_r)r&=0,\end{aligned}$$

where $k_A^2=A/m$, $k_E^2=E/m$, $k_c^2=C/m$.

These are the fundamental equations with which we have to deal. If we had assumed that the steady motion had been along a line making a constant angle with the horizon, slightly different equations would be obtained.*

**113.** The equations that we have to solve are linear equations with constant coefficients. The solution is most readily accomplished by means of symbolic operators.† Consider first the equations for longitudinal motion. Let

$$\Delta=\begin{vmatrix} D-X_u, & -X_w & -(X_q D+g)\\ -Z_u, & D-Z_w, & -(Z_q+U)D\\ -M_u, & -M_w, & (k_B^2D^2-M_q D)\end{vmatrix}.$$

The determinant on the right is to be developed according to the ordinary method, the symbol $D$ representing differentiation. When this is done we find

$$\Delta=AD^4+BD^3+CD^2+DD+E,$$

* Bairstow, loc. cit., Bryan, Chapter I, Cowley and Evans, Chapter XI.

† Murray's "Differential Equations," Chapter XI. Wilson's "Advanced Calculus," p. 223.

where $A$, $B$, $C$, $D$, $E$ are constants, whose explicit values will be given presently. The quantities $u$, $w$, $\theta$ are then all solutions of the single equation of the fourth order,

$$AD^4\xi+BD^3\xi+CD^2\xi+DD\xi+E=0.$$

In a similar way we put

$$\Delta'=\begin{vmatrix} D-Y_v, & -(Y_pD-g), & U-Y_r, \\ -L_v, & (k_A^2D^2-L_pD) & -(k_E^2D+L_r) \\ -N_v, & -(k_E^2D^2+N_pD), & (k_c^2D-N_r) \end{vmatrix},$$

which gives, upon development,

$$\Delta'=A_1D^4+B_1D^3+C_1D^2+D_1D+E_1.$$

The quantities $v$, $\phi$, $r$ are then all solutions of

$$A_1D^4\xi+B_1D^3\xi+C_1D^2\xi+D_1D\xi+E_1=0.$$

The conditions for stability can now be given in a general way. The quantities $u$, $w$, $\theta$ will be linear combinations of the form

$$ae^{\lambda_1 t}+be^{\lambda_2 t}+ce^{\lambda_3 t}+de^{\lambda_4 t},$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ are the four roots of the quartic

$$A\lambda^4+B\lambda^4+C\lambda^2+D\lambda+E=0,$$

and $v$, $\phi$, $r$ will be of the same form, where $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ are roots of

$$A_1\lambda^4+B_1\lambda^3+C_1\lambda^2+D_1\lambda+E_1=0.$$

In order that the machine be stable it is therefore necessary that the roots of the two quartic equations have their real parts negative. For if this condition is fulfilled the values of $u$, $v$, $w$, $\theta$, $\phi$, $r$ become smaller and smaller as the time increases; they thus approach zero, and a steady motion is therefore resumed. If, on the other hand, the real part of one of the roots is positive, the term corresponding to it increases and instability will result. It is to be noted that a machine can have longitudinal stability, but asymmetric instability, or vice versa. (We shall, however, see later that asymmetric stability is the more difficult to obtain.)

**114.** The necessary condition that the real roots be negative, and the real parts of the imaginary roots be negative in a quartic equation is given by Routh.* Let the equation be

$$ax^4+bx^3+cx^2+dx+e=0.$$

Then the roots will have the property mentioned if, and only if, the coefficients

$$a,\ b,\ c,\ d,\ e,$$

and the discriminant, called Routh's discriminant,

$$bcd-ad^2-eb^2$$

are all positive,

An airplane will therefore be stable longitudinally if

$$A,\ B,\ C,\ D,\ E,\quad \text{and}\quad BCD-AD^2-EB^2$$

are all positive.

It will be asymmetrically stable if

$$A_1,\ B_1,\ C_1,\ D_1,\ E_1,\quad \text{and}\quad B_1C_1D_1-A_1D_1^2-E_1B_1^2$$

are all positive.

**115.** If the determinant $\Delta$ be developed, we find:

$$A=k_B^2,\qquad B=-(M_q+X_uk_B^2+Z_wk^2),$$

$$C=\begin{vmatrix} Z_w, & U+Z_q \\ M_w, & M_q \end{vmatrix}+\begin{vmatrix} X_u, & X_q \\ M_u, & M_q \end{vmatrix}+k_B^2\begin{vmatrix} X_u, & X_w \\ Z_u, & Z_w \end{vmatrix}$$

$$D=-\begin{vmatrix} X_u, & X_w, & X_q \\ Z_u, & Z_w, & U+Z_q \\ M_u, & M_w, & M_q \end{vmatrix}-gM_u,$$

$$E=-g\begin{vmatrix} Z_u, & Z_w \\ M_u, & M_w \end{vmatrix}.$$

In the same way, by developing $\Delta'$ we obtain:

$$A_1=\begin{vmatrix} k_A^2, & -k_E^2 \\ -k_E^2, & k_c^2 \end{vmatrix},$$

* "Stability in Motion." p. 14.

$$B_1 = Y_v \begin{vmatrix} -k_A^2, & -k_E^2 \\ k_E^2, & k_C^2 \end{vmatrix} - \begin{vmatrix} k_E^2, & L_r \\ -k_E^2, & N_r \end{vmatrix} - \begin{vmatrix} L_p, & -k_E^2 \\ N_p, & k_C^2 \end{vmatrix},$$

$$C_1 = \begin{vmatrix} Y_v, & Y_p, & 0 \\ L_v, & L_p, & -k_E^2 \\ N_v, & N_p, & k_E^2 \end{vmatrix} - \begin{vmatrix} Y_v, & 0, & Y_r - U \\ L_v, & -k_A^2, & L_r \\ N_v, & k_E^2, & N_r \end{vmatrix} + \begin{vmatrix} L_p, & L_r \\ N_p, & N_r \end{vmatrix},$$

$$D_1 = -\begin{vmatrix} Y_v, & Y_p, & Y_r - U \\ L_v, & L_p, & L_r \\ N_v, & N_p, & N_r \end{vmatrix} + g \begin{vmatrix} L_v, & -k_E^2 \\ N_v, & k_C^2 \end{vmatrix},$$

$$E_1 = -g \begin{vmatrix} L_v, & L_r \\ N_v, & N_r \end{vmatrix}.$$

**116.** The whole question of determining the stability of a machine depends upon the determination of the resistance derivatives, the knowledge of which will give the coefficients in the two quartics. This is a practical question which is solved by experiments on a model in a wind tunnel. We shall give only a slight indication of the procedure. The question is discussed at length by Bairstow.*

Consider the resistance derivatives that depend on $u$. By means of instruments which hold the model in the wind tunnel, the resistance $X_0$ for the velocity $U$ is determined, the $x$-axis being in the direction of the wind. We assume

$$X = cu^2,$$

where $c$ is a constant and $u$ the velocity. Hence

$$\frac{\partial X}{\partial u} = X_u = 2cu = 2u \cdot \frac{X_0}{u^2} = 2\frac{X_0}{u}.$$

* Loc. cit., pp. 154–158. See also: Hunsaker, "Experimental Analysis of Inherent Longitudinal Stability for a Typical Biplane." First Annual Report of the (American) National Advisory Committee for Aeronautics, Hunsaker, "Dynamical Stability of Aeroplanes," Smithsonian Miscellaneous Collection, Vol. 62, No. 5. These two papers by Hunsaker will be referred to as Hunsaker (1), Hunsaker (2), respectively. The second paper especially gives a full discussion of the subject referred to here.

In the same way $Z_u$, $M_u$, $N_u$ are found from a knowledge of $Z_0$, $M_0$.

The derivatives that depend upon $v$ are found by turning the model about the $z$-axis through some fixed angle, for the effect of such a turning is to give a component of the air reaction parallel to the $y$-axis. In the same way the derivatives dependent on $w$ are found by making measurements after the machine is turned about the $y$-axis through some angle.

The derivatives that depend upon $p$, $q$, $r$ are termed rotary derivatives by Bryan. They are determined by oscillating the machine about the various axes, controlling the oscillations by means of springs, and measuring the damping by means of photography.

A discussion of the range in numerical value that may be expected in the resistance derivatives for machines generally in use will also be found in the report given by Bairstow.

**117.** As an example of the numerical values, we take those given by Bairstow for a typical machine. They are:

| | | |
|---|---|---|
| $k_B^2 = 25$ | $m = 40$ slugs * | |
| $X_u = -.14$ | $Z_u = -.80$ | $M_u = 0$ |
| $X_w = .19$ | $Z_w = -2.89$ | $M_w = 2.66$ |
| $X_q = .5$ | $Z_q = 9$ | $M_q = -210$ |
| $k_A^2 = 25$ | $k_C^2 = 35$ | $k_E^2 = 0$ |
| $Y_v = -.25$ | $L_v = .83$ | $N_v = .54$ |
| $Y_p = 1$ | $L_p = -200$ | $N_p = 28$ |
| $Y_r = -3$ | $L_r = 65$ | $N_r = .37$ |

The equation for longitudinal stability is

$$\lambda^4 + 11.4\lambda^3 + 33.6\lambda^2 + 5.72\lambda + 2.72 = 0,$$

and for lateral stability

$$\lambda^4 + 9.31\lambda^3 + 9.81\lambda^2 + 10.15\lambda - .161 = 0.$$

* The engineering unit of mass, equal to 32.16 pounds. It is used here in order to have consistent units. See appendix for a further discussion.

The Routh discriminant for the first equation is found to be positive. Therefore the machine possesses longitudinal stability. The negative term in the second equation shows the machine to be asymmetrically unstable. Disturbances therefore would need to be corrected by the assistance of the controls. Bairstow shows that the machine becomes asymmetrically stable when gliding at a glide of 1:6 with the propeller cut off. To prove this it is merely necessary to have assumed that the steady motion had been in a line inclined at some angle to the horizon. The development is similar to that which has been given here.

**118.** The stability of a machine will depend upon its speed, and a machine may be stable at one speed and unstable at another. This is due to variation of the various resistance derivatives. Thus, while it is possible to make a machine with a considerable speed range through a change of elevator setting, the machine cannot be expected to be uniformly stable at all the speeds at which it may fly. At high speed the pilot may be able to abandon his controls, the machine possessing so great an inherent stability as to be able to "fly itself," while at low speeds constant watchfulness and attention may be necessary. An interesting example of this sort is given by Hunsaker.* In his report the longitudinal stability of a machine is investigated for speeds varying from 79 miles an hour, corresponding to an angle of attack of 1°, to a speed of 43.7 miles an hour, corresponding to an angle of attack of 15°.5. Instability at low speeds corresponding to an angle of attack larger than 10°.5 results from the Routh discriminant becoming negative.†

**119.** The general discussion of the quartic equations under consideration presents considerable difficulty. Bairstow has given an approximate factorization of the equations.‡ That is, he has given factors that, while not algebraically exact, give a fairly close approximation when the numerical values of the coefficients are used. And from the factors he gives a general

* (1) Loc. cit., pp. 47–51.

† Hunsaker (2), p. 65, considers the lateral stability of a Clark tractor and shows it is laterally stable except at low speed.

‡ Loc. cit.

idea of the properties of the plane that contribute to its stability.

The equation for longitudinal stability Bairstow factors into

$$\left(\lambda^2+\frac{B}{A}\lambda+\frac{C}{A}\right)\left(\lambda^2+\left[\frac{D}{C}-\frac{BE}{C^2}\right]\lambda+\frac{E}{C}\right)=0.$$

In general, it may be said that the first factor represents a very short oscillation, which in the majority of machines will die out quite rapidly. The second factor, however, represents a relatively long oscillation, which is dependent in its nature upon the speed, and may cause instability at the low speeds.

The factored form for the equation for lateral stability is

$$\left(\lambda+\frac{E_1}{D_1}\right)\left(\lambda+\frac{B_1^2-A_1C_1}{A_1B_1}\right)\left(\lambda^2+\left[\frac{C_1}{B_1}-\frac{E_1}{D_1}\right]\lambda+\frac{B_1D_1}{B_1^2-A_1C_1}\right)=0.$$

It can be shown in general that the real root corresponding to the second factor is negative, and that the real parts of those corresponding to the third factor are negative. Thus these factors indicate stability. A consideration of the first factor shows that stability requires that $E_1$ and $D_1$ be of the same sign. It is found that $D_1$ can be universally regarded as being positive. Thus $E_1$ must be rendered positive. This condition is the most difficult to obtain in construction.*

**120.** As an example of the accuracy of the approximate factorization we shall consider one of the cases considered by Hunsaker.† The data obtained by experiments are:

$i$ = angle of attack = 1°.

Velocity = 79 miles an hour, $U=-115.5$ foot-seconds.

$m=55.9$ slugs, $\quad K_B=34.$

$X_u=-.128, \quad X_w=\ .162, \quad M_w=\ 1.74$

$Z_u=-.557, \quad Z_w=-3.95, \quad M_q=-150.$

This gives $A=34$, $B=289$, $C=834$, $D=115$, $E=31$, $BCD-AD^2-B^2E=18\times10^6$. Hence the machine is longitudinally stable.

* For further discussions of the influence of design, and general consideration of this question, see Bairstow, loc. cit. Hunsaker (1), (2.) Cowley and Evans, Chap. XII.

† (1). Article 13, Case I.

The equation for longitudinal oscillations is

$$34\lambda^4+289\lambda^3+834\lambda^2+115\lambda+31=0.$$

The approximate factorization from Bairstow's formulæ is

$$(\lambda^2+8.5\lambda+24.5)(\lambda^2+.125\lambda+.0374)=0.$$

The roots obtained from the first factor are

$$\lambda=-4.25\pm 2.54i,$$

which give the short oscillation of period $\frac{2\pi}{2.54}=2.5$ seconds.

The roots obtained from the second factor are

$$\lambda=-.063\pm .183i,$$

which give the long oscillations of period 34.3 seconds.

More accurate values of the roots of the quartic for the same machine are given by Wilson,* and are

$$\lambda=-4.180\ \pm 2.430i,$$

$$\lambda=-\ .0654\pm\ .187i.$$

**121. Effect of Gusts.**—We have so far considered the disturbance occurring while the machine is in still air. As the atmosphere is in continual and irregular movement the effect of gusts must be considered. A rather comprehensive study of this nature has been made by Wilson.* The actual nature of gusts that occur in the air can obviously not be represented by mathematical means, because of our lack of knowledge of the exact conditions and fluctuations in the atmosphere. However, we can assume certain "mathematical gusts" which would seem to have as unstabilizing an effect on an airplane as an actual gust, and by becoming assured of a satisfactory behavior of the machine in such mathematical gusts of widely different nature, we can regard the machine as a vehicle possessing an air-worthiness sufficient for the conditions which it may

* E. B. Wilson, "Theory of an Aeroplane Encountering Gusts." First Annual Report of the (American) Advisory Committee on Aeronautics, Report No. 1, part 2, p. 58. The quartic given by Wilson is

$$34\lambda^4+288.7\lambda^3+833.0\lambda^2+115.1\lambda+31.18=0.$$

in practice be expected to encounter. The equations that have been developed in what precedes give the basis for an investigation of the nature mentioned.

We shall merely indicate the method, without going into the details of the development, for considerable calculation is involved. Reference can be made to the detailed and clear report by Wilson.

**122.** Suppose first that the machine is flying horizontally and encounters a gust moving parallel to the $x$-axis. We think of it as being caused by a motion of the air parallel to the $x$-axis, with veolocity $u_1$, which we shall take to be positive when along the negative $x$-axis. A displacement of the machine results, and the gust is no longer directly along the $x$-axis. But assuming that all displacements are relatively small, we shall say that the gust continues unchanged along the $x$-axis. The machine has an altered velocity along the $x$-axis itself, of amount $u$, according to our former notation. Therefore the total change in relative wind is $u+u_1$. This is consequently the quantity by which we should multiply the resistance derivative $X_u$ in the equations of motion.

In the general case we shall assume that the gust has components $u_1$, $v_1$, $w_1$, $p_1$, $q_1$, $r_1$. The equations for the longitudinal motion will then be obtained by altering, as indicated above, the quantities by which the various resistance derivatives are multiplied. We have then for the equations of motion:

$$(D-X_u)u-X_w w-(X_q D+g)\theta = X_u u_1+X_w w_1+X_q q_1,$$
$$-Z_u u+(D-Z_w)w-(Z_q+D)D\theta = Z_u u_1+Z_w w_1+Z_q q_1,$$
$$-M_u u-M_w w+(k_B{}^2D^2-M_q D)\theta = M_u u_1+M_w w_1+M_q q_1.$$

In these equations, $u_1$, $v_1$, . . . . $r_1$ are supposed to be known functions of the time which vanish for the moment when the gust commences. The solution will consist of the complimentary function and the particular integrals. The first step is to solve the system of equations algebraically. We shall take the result for $u$ as typical. We find

$$\Delta u = \Delta_1 u_1+\Delta_2 w_1+\Delta_3 q_1,$$

where $\Delta$ is the determinant used before, and $\Delta_1$, $\Delta_2$, $\Delta_3$, are determinants obtained from $\Delta$ by replacing the elements of the first column by certain of the resistance derivatives. These determinants contain the operator $D$; they are to be expanded as though $D$ were an algebraic quantity, and then applied as operators to the known functions $u_1$, $w_1$, $q_1$. The form of the determinants makes a general development very difficult. For a particular machine the knowledge of the resistance derivatives allow the operators $\Delta_1$, $\Delta_2$, $\Delta_3$ to be completely determined. We can then write as a final equation,

$$\Delta u = \bar{u}(t),$$

where $\bar{u}(t)$ is a known function of $t$ as soon as we have assumed a character for the gust. Similar equations exist for $w$ and $\theta$. Represent the particular integrals by $I_u$, $I_w$, $I_\theta$, respectively. We then have for the final solutions of the equations:

$$u = c_{11}e^{\lambda_1 t} + c_{12}e^{\lambda_2 t} + c_{13}e^{\lambda_3 t} + c_{14}e^{\lambda_4 t} + I_u,$$

$$w = c_{21}e^{\lambda_1 t} + c_{22}e^{\lambda_2 t} + c_{23}e^{\lambda_3 t} + c_{24}e^{\lambda_4 t} + I_w,$$

$$\theta = c_{31}e^{\lambda_1 t} + c_{32}e^{\lambda_2 t} + c_{33}e^{\lambda_3 t} + c_{34}e^{\lambda_4 t} + I_\theta,$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ are the roots of the quartic that gives the longitudinal motion, and the quantities $c_{ij}$ are constants. These constants must be determined so that $u$, $w$, $\theta$ all vanish for $t=0$, the time at which the gust commences. The fact that the expressions obtained on dropping $I_u$, $I_w$, $I_\theta$, must satisfy the differential equations enables us in the first place to determine the ratios $c_{1i} : c_{2i} : c_{3i}$ $(i=1, 2, 3, 4)$. The solutions can then be expressed in terms of $c_{11}$, $c_{12}$, $c_{13}$, $c_{14}$. The values of these are to be determined so that $u$, $w$, $\theta$, $q=d\theta/dt$ are all zero for $t=0$. This gives us four equations involving the constants and the four quantites $I_{u0}$, $I_{w0}$, $I_{\theta 0}$, $I'_{\theta 0}$, which are the values of the particular integrals for $t=0$. While imaginary quantities occur in the process of the work, the final result can be put in a trigonometric form free from imaginaries. The details of this development are given in Wilson's report, articles 2, 3, 4.

**123.** As an illustration of the application of the method we shall take the first gust treated by Wilson, for the machine referred to in § 120. Let the gust be head-on, and represented by

$$u_1 = J(1 - e^{-.2t}), \qquad w_1 = q_1 = 0.$$

It is seen that the gust increases slowly from 0 to $J$ in intensity. The values of the particular integrals found by Wilson are:

$$I_u = -J(1 - .247e^{-.2t}), \qquad \therefore\ I_{u0} = -.753J,$$
$$I_w = .082Je^{-.2t}, \qquad I_{w0} = -.082J,$$
$$I_\theta = -.00495Je^{-.2t}, \qquad I_{\theta 0} = -.0049J,$$
$$I_\theta' = .00099Je^{-.2t}, \qquad I_{\theta 0}' = .00099J.$$

The complete solution of the differential equations then gives:

$$u = Je^{-.0654t}(.622 \cos .187t + .630 \sin .187t)$$
$$J(1 - .247e^{-.2t}),$$
$$w = Je^{-4.18t}(-.004 \cos 2.43t + .003 \sin 2.43t)$$
$$-Je^{-.0654t}(.078 \cos .187t + .059 \sin .187t) + .082Je^{-.2t},$$
$$\theta = Je^{-.0654t}(.00495 \cos .187t - .0031 \sin .187t) - .00495Je^{-.2t}.$$

The effect of the gust is given by Wilson as follows: "(1) The machine takes up an easy slowly damped oscillation in $u$ of amplitude about 89 per cent of $J$; after the oscillation dies out the machine is making a speed $J$ less relative to the ground and hence the original relative speed to the wind. (2) There is a rapidly damped oscillation in $w$ of rather small magnitude and a slowly damped one of about 10 per cent of $J$, the final condition being that of horizontal flight. (3) There is a slow oscillation in pitch of about $.0058J$ radians, or about $.32J°$. If the magnitude of $J$ is great, the pitching becomes so marked that the approximate method of solution can no longer be considered valid—a gust of 20 foot-seconds causing a pitch of some 6°. As the period is long (about one-half minute), the pilot should have ample time to correct the trouble before it produces serious consequences."

**124.** During the interval that elapses between the commencement of the gust and the acquirement of the subsequent steady motion the altitude of the machine changes. It is possible to determine this change. Let the vertical velocity be represented by $\frac{d\xi}{dt}$, $\xi$ being measured upwards. Then, by resolving $u$ and $w$ along the vertical we have

$$\frac{d\xi}{dt}=w\cos\theta+(V+u)\sin\theta.$$

Hence, approximately, the change in altitude is

$$\xi=\int_0^t(w+V\theta)dt.$$

For the machine and gust considered this becomes

$$\xi=J\int_0^t[e^{-.0654t}(.5\cos .187t-.4\sin .187t)-.5e^{-.2t}]dt$$

$$=J[e^{-.0654t}(\cos .187t+3\sin .187t)+2.5e^{-.2t}-3.5].$$

When the steady motion has been reached the limit of this is $-3.5J$. For a head-on gust $J$ is negative. Hence the machine would rise 70 feet on encountering a head-on gust of magnitude 20 foot-seconds.

# APPENDIX

## I

**1.** In the fundamental equation,

$$F = kAV^2,$$

that we have used for the air pressure, the value of the constant $k$ depends upon the units we are using. Thus for pressure on a flat plane, we have:

$F = .0030AV^2$, if $A$ is in sq. ft., $V$ in mi. per hr., $F$ in lbs.

$F = .00143AV^2$, if $A$ is in sq. ft., $V$ in ft. per sec., $F$ in lbs.

$F = .075AV^2$, if $A$ is in sq. m., $V$ in m. per sec., $F$ in kg.

$F = .0057AV^2$, if $A$ is in sq. m., $V$ in km. per hr., $F$ in kg.

In case we wish to compare the results that different experimenters obtain for certain constants, it is necessary to properly take account of the units employed, before an opinion can be formed as to the agreement of the results. Similarly, if we wish to adapt to English units the lift and drag coefficients for a wing which has been tested in a wind tunnel where metric units are employed, it is necessary to have a ready means of making the required transformations.

**2.** Let us write

$$F = C\rho AV^2,$$

where $\rho$ denotes the density of the air. The dimensions of $\rho$, $A$, $V^2$ are respectively, $ML^{-3}$, $L^2$, $L^2T^{-2}$. Therefore the dimensions of $\rho AV^2$ are $MLT^{-2}$. But the dimensions of force, $F$, are also $MLT^{-2}$. Consequently the constant $C$ is dimensionless, that is, it is an *absolute* constant, independent of the units

employed, provided the system of units is self consistent, that is, the unit of force is the force required to give to a unit of mass a unit velocity in a unit of time. Therefore through the use of the air density we can compare the results of experiments conducted in different units, or can readily adapt to one system a series of measurements made in another system of units. We need merely write the constant $k$ ($K_v$ or $K_z$), used before, in the form,

$$k = C\rho,$$

where $\rho$ is the density of the air in the units employed.

**3.** We shall take some examples.

(1) Take the meter as unit of length, the second as unit of time, and the weight of a kilogram as the unit of force. In order to have consistent units, the unit of mass is then 9.8 kilograms. The density in C.G.S units of dry air at 15.6° C. (60° F.), and pressure of 76 cm. of mercury is .001225. Hence the mass of a cubic meter is $.001225 \times 10^6/10^3 = 1.255$ kilograms. In the system we have adopted we have therefore for the density $\rho = 1.225/9.8 = .125$. Therefore,

$$k = .125C, \quad \text{or} \quad C = 8k.$$

In § 1 we had $k = .075$ for the units employed here. Hence $C = .600$.

(2) Take the foot as a unit of length, the second as unit of time, and the pound as unit of force. The unit of mass is 32.2 lbs. (a slug). The density of the air is now .00238. Hence

$$k = .00238C, \quad \text{or} \quad C = 420.2k.$$

In § 1 we had $k = .00143$ for the units used here. Hence $C = .601$, which agrees with the results given above for metric units.

(3) Take the same units for force and length as in (1), but take the kilometer per hour as the unit of velocity. (This is not a consistent system of units.)

We write

$$F = k'A(V \text{ km./hr.})^2.$$

But

$$F = .125CA(V \text{ m./sec.})^2$$
$$= .125CA(1000/3600 \times V \text{ km./hr.})^2$$
$$= .0096CA(V \text{ km./hr.})^2.$$

Hence

$$k' = .0096C \quad \text{or} \quad C = 104.2k'.$$

If we take $C = .600$ we have $k' = .0057$, which is the coefficient given in § 1 for the units in question.

(4) Take the same units as in (2), except that we express velocities in miles per hour.

We write

$$F = k'A(V \text{ mi./hr.})^2.$$

But

$$F = .00238CA(V \text{ ft./sec.})^2$$
$$= .00238CA(5280/3600 \times V \text{ mi./hr.})^2$$
$$= .00512CA(V \text{ mi./hr.})^2.$$

Hence

$$k' = .00512C, \quad \text{or} \quad C = 195.3k'.$$

Taking $C = .600$ we find $k' = .0030$, which agrees with the value given in § 1.

As an illustration of the use of the results obtained we take the following problem. The lift coefficient on a certain wing at an angle of attack of 4° is .001455, the units being the pound, the square foot, and mi./hr. What is the value of the coefficient if the units are the kilogram, the square meter, and the meter per second?

We have for the value of the absolute constant by (4), (for this wing at this angle of attack),

$$C = 195.3 \times .001455 = .2841.$$

By (1) the value of the lift coefficient in the new units would be

$$.2841/8 = .0355.$$

(To make a change of the sort considered here we evidently multiply by $195.3/8 = 24.4$. All types of changes of units

likely to occur can be similarly easily obtained from the results above.)*

## II

4. In considering flight at different altitudes it has been necessary to consider the varying density of the air. For known conditions at the surface, that is, known surface pressure and temperature, the pressure, density, and temperature at any altitude can be calculated. The formulæ for this purpose can be found in texts on physics, or more extended aëronautic treatises. We shall merely give a table showing the variation for certain assumed surface conditions. The ratio $\mu(z)$ is that of the pressure at altitude $z$ to the pressure at the surface of the earth, and the quantity $\rho(z)$ is the ratio of the air density at altitude $z$ to the surface density. It will be seen that for moderate altitudes these two quantities are approximately equal.

| Altitude, Feet. | Temp., Fahr. | Pressure. | | Density. | |
|---|---|---|---|---|---|
| | | Inches, Mercury. | Ratio, $\mu(z)$. | Slugs. | Ratio, $\rho(z)$. |
| 0 | 48° | 30.0 | 1.000 | .00246 | 1.000 |
| 2,000 | 44 | 28.1 | 0.937 | .00232 | .940 |
| 4,000 | 41 | 26.2 | .873 | .00215 | .875 |
| 6,000 | 39 | 24.4 | .813 | .00201 | .815 |
| 8,000 | 36 | 22.6 | .753 | .00187 | .760 |
| 10,000 | 30 | 20.9 | .696 | .00175 | .710 |
| 12,000 | 25 | 19.3 | .645 | .00163 | .665 |
| 14,000 | 20 | 17.9 | .596 | .00151 | .635 |
| 16,000 | 12 | 16.5 | .550 | .00144 | .600 |
| 18,000 | 3 | 15.2 | .508 | .00133 | .540 |
| 20,000 | −5 | 14.0 | .467 | .00125 | .510 |
| 22,000 | −14 | 13.0 | .432 | .00118 | .480 |
| 24,000 | −23 | 12.0 | .400 | .00108 | .440 |

* For a discussion of the question of units see Everett's "Illustrations of the C. G. S. System of Units with Tables of Physical Constants," Macmillan & Co., London and New York, 4th edition, 1891.

## III

**5.** The following table is given for the purpose of changing velocities from miles per hour to feet per second, and conversely.

| mi./hr. | ft./sec. | ft./sec. | mi./hr. |
|---|---|---|---|
| 30 | 44.00 | 50 | 34.09 |
| 40 | 58.66 | 60 | 40.90 |
| 50 | 73.32 | 70 | 47.72 |
| 60 | 88.00 | 80 | 54.54 |
| 70 | 102.66 | 90 | 61.37 |
| 80 | 117.32 | 100 | 68.18 |
| 90 | 132.00 | 110 | 74.99 |
| 100 | 146.66 | 120 | 81.81 |
| 110 | 161.36 | 130 | 88.62 |
| | | 140 | 95.44 |
| | | 150 | 102.26 |

## IV

**6.** *References.*—The literature on the subject of Aviation has grown with great rapidity in the past few years. Although no attempt at completeness is made in the following list of references it is hoped that the different aspects of the subject are adequately covered. The title of a work indicates whether it is of general or special character.

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